SOME IMPLICATIONS OF THE THEORY OF WEAVES WITH APPLICATIONS IN DETERMINATENESS AND LOGIC

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THESIS

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SECTION 1: INTRODUCTION

The notion of a "weave" was defined by Gaisi Takeuti as an approach to the problem of the determinateness of games (among other things). It was suspected that weaves would be interesting in their own right as set-theoretic structures. The discovery of The Fundamental Theorem of Normal Weaves indicated the truth of these suspicions. In this thesis we present still more evidence.

We define the term "normality" for weaves. Roughly speaking, normality is analoguous to the notion of determinateness for games. The Fundamental Theorem tells us that normal weaves can be viewed as combinations of other normal weaves, and that the methods of combination bear similarity to the logical connectives $\wedge$ and $\vee$. We use this knowledge in Sections 3 and 4 to give two ways of representing weaves. In Sections 5 through 8 we present and clarify some of the properties of weaves. In Section 10 we present still another way to represent weaves. This representation is quite different from the ones given in Sections 3 and 4, and does not make explicit use of The Fundamental Theorem (although The Fundamental Theorem is still used to prove theorems <u>about</u> this method of representation). Sections 9 and 11 are devoted specifically to presenting two problems concerning weaves. We present some possible directions for the solutions of these problems. In Sections 12 and 13 we exploit the similarity between the logical connectives $\wedge$ and $\vee$ and the methods of combining weaves. We use weaves to define two non-standard systems of logic.

We begin with a definition.

DEFINITION: Let D be a set. Let $\mathcal{L}$ and $\mathcal{R}$ be non-empty subsets of $\mathcal{P}(D) - \{\emptyset\}$. The pair $<\mathcal{L}, \mathcal{R}>$ is a <u>weave</u> of D iff

(1) $\forall L \in \mathcal{L} \quad \forall R \in \mathcal{R} \quad L \cap R$ is a singleton set, and

(2) $\forall d \in D \quad \exists L \in \mathcal{L} \quad \exists R \in \mathcal{R}$ s.t. $\{d\} = L \cap R$.

The set D is called the <u>alphabet</u> of $<\mathcal{L}, \mathcal{R}>$, and we denote this by writing $D = \text{Alph}(<\mathcal{L}, \mathcal{R}>)$. More generally, if $\mathcal{L}$ is any non-empty family of sets, let Alph($\mathcal{L}$) be the union of all sets in $\mathcal{L}$. Elements of Alph($\mathcal{L}$) are called <u>letters</u>.

EXAMPLES:

(1) $\mathcal{L} = \{\{0\}, \{1\}, \{2\}\}$

$\mathcal{R} = \{\{0, 1, 2\}\}$

The pair $<\mathcal{L}, \mathcal{R}>$ is a weave of $\{0,1,2\}$.

(2) $\mathcal{L} = \{\{a,d\}, \{b,c\}\}$

$\mathcal{R} = \{\{a,b\}, \{c,d\}, \{a,c\}, \{b,d\}\}$

(3) $\mathcal{L} = \{\{a,d\}, \{b,c\}\}$

$\mathcal{R} = \{\{a,b\}, \{c,d\}, \{a,c\}\}$

DEFINITION: The pair $< \mathcal{L}, \mathcal{R} >$ is called a trivial weave iff either

(a) $\mathcal{L} = \{\{d\} | d \in D\}$ and $\mathcal{R} = \{D\}$, or

(b) $\mathcal{L} = \{D\}$ and $\mathcal{R} = \{\{d\} | d \in D\}$.

A very important property of weaves is the property of being normal.

DEFINITION: The weave $< \mathcal{L}, \mathcal{R} >$ is normal iff $\forall X \subseteq Alph(< \mathcal{L}, \mathcal{R} >)$ either $\exists L \in \mathcal{L}$ s.t. $L \subseteq X$ or $\exists R \in \mathcal{R}$ s.t. $R \subseteq Alph(< \mathcal{L}, \mathcal{R} >)-X$.

Notice that example (2) is a normal weave, but example (3) is not, because X can equal {a,c}.

It is difficult to get a useful intuitive feeling about what "normality" really means. The following set of lemmas should help.

DEFINITION: Let $\mathcal{L}$ be a non-empty family of sets. The set C is a choice set for $\mathcal{L}$ (abbreviated, cs for $\mathcal{L}$) iff $C \subseteq Alph(\mathcal{L})$ and, for each set L in $\mathcal{L}$, $C \cap L \neq \emptyset$. The set G is a good choice set for $\mathcal{L}$ (abbreviated, gcs for $\mathcal{L}$) iff $G \subseteq Alph(\mathcal{L})$ and, for each set L in $\mathcal{L}$, $G \cap L$ is a singleton set.

The family of choice sets for $\mathcal{L}$ is denoted $\mathcal{C}(\mathcal{L})$. The family of good choice sets for $\mathcal{L}$ is denoted $\mathcal{G}(\mathcal{L})$.

Notice that a cs or a gcs for $\mathcal{L}$ must be a subset of $Alph(\mathcal{L})$. This subtlety will become an important issue in the proof of The Fundamental Theorem.

LEMMA 1.1: If A and B are both gcs's for $\mathcal{L}$, and $A \subseteq B$, then A = B.

PROOF: Assume $b \in B - A$. Since $B \subseteq Alph(\mathcal{L})$, there is a set L in $\mathcal{L}$ such that $b \in L$. So $B \cap L = \{b\}$. But then, since $b \notin A$, we have $A \cap L = \emptyset$, contradicting the fact that A is a gcs for $\mathcal{L}$. Q.E.D.

If < $\mathcal{L}$, $\mathcal{R}$ > is a weave, then every set L in $\mathcal{L}$ is a gcs for $\mathcal{R}$, and every set R in $\mathcal{R}$ is a gcs for $\mathcal{L}$, so we have the following.

COROLLARY: If R_1, R_2 ε $\mathcal{R}$ and $R_1 \subseteq R_2$, then $R_1 = R_2$. If L_1, L_2 ε $\mathcal{L}$ and $L_1 \subseteq L_2$, then $L_1 = L_2$.

LEMMA 1.2: If < $\mathcal{L}$, $\mathcal{R}$ > is normal, then

(1) $\mathcal{R} = \mathcal{G}(\mathcal{L})$, and

(2) $\mathcal{L} = \mathcal{G}(\mathcal{R})$.

PROOF: To prove (1) it is enough to show that, for every G which is a gcs of $\mathcal{L}$, G is in $\mathcal{R}$.

Let G be a gcs of $\mathcal{L}$. Then there is no set L in $\mathcal{L}$ such that $L \subseteq$ Alph($\mathcal{L}$) - G (because otherwise L would not meet G), so, by normality, there is a set R in $\mathcal{R}$ such that $R \subseteq G$. But R and G are both gcs's of $\mathcal{L}$, so by Lemma 1.1, R = G. Q.E.D.

We often abbreviate the notation for a weave < $\mathcal{L}$, $\mathcal{R}$ >, where $\mathcal{R} = \mathcal{G}(\mathcal{L})$, by simply writing $\mathcal{L}$. If we write " $\mathcal{L}$ is normal", we mean that we have a normal weave < $\mathcal{L}$, $\mathcal{R}$ >, in which case $\mathcal{R} = \mathcal{G}(\mathcal{L})$ and $\mathcal{L} = \mathcal{G}(\mathcal{R})$.

LEMMA 1.3: The weave < $\mathcal{L}$, $\mathcal{R}$ > is normal iff, for every X, if X is a cs of $\mathcal{L}$, there exists an $X' \subseteq X$ such that X' is a gcs of $\mathcal{L}$.

PROOF: The formula

$$\forall X\ (X \text{ is a cs of } \mathcal{L} \Rightarrow \exists X' \subseteq X \text{ s.t. } X' \text{ is a gcs of } \mathcal{L})$$

is true iff

$$\forall A\ (\bar{A} \text{ is a cs of } \mathcal{L} \Rightarrow \exists R \subseteq \bar{A} \text{ s.t. } R \text{ is a gcs of } \mathcal{L})$$

(where $\bar{A}$ is Alph($\mathcal{L}$)-A). This last formula is true in turn iff

$$\forall A\ (\neg \exists L \in \mathcal{L} \text{ s.t. } L \subseteq A \Rightarrow \exists R \subseteq \bar{A} \text{ s.t. } R \text{ is a gcs of } \mathcal{L}),$$

which is true iff

$$\forall A\ (\neg \exists L \in \mathcal{L} \text{ s.t. } L \subseteq A \Rightarrow \exists R \subseteq \bar{A} \text{ s.t. } R \in \mathcal{R}),$$

which is true iff

$$\forall A (\exists L \in \mathcal{L} \text{ s.t. } L \subseteq A \vee \exists R \in \mathcal{R} \text{ s.t. } R \subseteq \bar{A})$$

is true. Q.E.D.

Lemma 1.3 will be very important in the next section.

LEMMA 1.4: For any non-empty family $\mathcal{L}$,

(1) $\mathcal{G}\mathcal{G}(\mathcal{L}) \supseteq \mathcal{L}$, and

(2) $\mathcal{G}\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{G}(\mathcal{L})$.

PROOF: (1) If $L \in \mathcal{L}$ and $G \in \mathcal{G}(\mathcal{L})$, then $L \cap G$ is a singleton, so $L \in \mathcal{G}\mathcal{G}(\mathcal{L})$. Thus $\mathcal{G}\mathcal{G}(\mathcal{L}) \supseteq \mathcal{L}$.

(2) By (1) we have $\mathcal{G}\mathcal{G}\mathcal{G}(\mathcal{L}) \supseteq \mathcal{G}(\mathcal{L})$. If $X \in \mathcal{G}\mathcal{G}\mathcal{G}(\mathcal{L})$ and

L ε $\mathcal{L}$, then X $\cap$ K is a singleton for every set K in $\mathcal{G}\mathcal{G}(\mathcal{L})$. But again by (1), L ε $\mathcal{G}\mathcal{G}(\mathcal{L})$. Thus X $\cap$ L is a singleton. So X ε $\mathcal{G}(\mathcal{L})$. Therefore $\mathcal{G}\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{G}(\mathcal{L})$. Q.E.D.

One of the reasons for studying weaves is that they resemble games. Questions about determinateness of games have their counterparts in questions about the normality of weaves, and vice-versa.

For instance, consider the well-known paper-scissors-stone game. Players I and II simultaneously present one of either "paper", "scissors", or "stone". (For definiteness we can declare player I to be the winner if both players present the same item.) This game can be represented by the following weave.

$$\mathcal{L} = \left\{ \begin{array}{l} \{\langle p,sc\rangle, \langle p,st\rangle, \langle p,p\rangle\} \\ \{\langle sc,sc\rangle, \langle sc,st\rangle, \langle sc,p\rangle\} \\ \{\langle st,sc\rangle, \langle st,st\rangle, \langle st,p\rangle\} \end{array} \right\}$$

$$\mathcal{R} = \left\{ \begin{array}{l} \{\langle p,sc\rangle, \langle sc,sc\rangle, \langle st,sc\rangle\} \\ \{\langle p,st\rangle, \langle sc,st\rangle, \langle st,st\rangle\} \\ \{\langle p,p\rangle, \langle sc,p\rangle, \langle st,p\rangle\} \end{array} \right\}$$

Here the set $\{\langle p,sc\rangle, \langle p,st\rangle, \langle p,p\rangle\}$ represents player I's presenting "paper". The set $\{\langle p,st\rangle, \langle sc,st\rangle, \langle st,st\rangle\}$ represents player II's presenting "stone".

The intersection of the two sets is <p,st>, meaning that player I picked "paper" and player II picked "stone", and so player I wins.

There is no winning strategy for either player in the paper-scissors-stone game, and notice that the weave $< \mathcal{L}, \mathcal{R} >$ is not normal. In fact, such a simultaneous-play game is determined iff the associated weave is normal.

What we want to do now is to connect this idea of determinateness of simultaneous-play games to that of determinateness for ordinary two person games as defined in [3]. Henceforth these ordinary two-person games will be referred to as <u>Gale-Stewart</u> games.

DEFINITION: Let $\mathcal{F} \subseteq \mathcal{P}(\mathrm{Alph}(\mathcal{L}))$. Then $< \mathcal{L}, \mathcal{R} >$ is called $\mathcal{F}$-<u>normal</u> iff, for every set X in $\mathcal{F}$, there exists an $L \in \mathcal{L}$ such that $L \subseteq X$, or there exists an $R \in \mathcal{R}$ such that $R \subseteq \mathrm{Alph}(\mathcal{L}) - X$.

DEFINITION: Let $\mathcal{F}_1 \subseteq \mathcal{P}(D_1)$ and $\mathcal{F}_2 \subseteq \mathcal{P}(D_2)$. A <u>tensor product</u> of $\mathcal{F}_1$ and $\mathcal{F}_2$, which is denoted $\mathcal{F}_1 \otimes \mathcal{F}_2$, is defined by the following:

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = \{\{< d_1, d_2 > | \, d_1 \in F_1 \wedge d_2 \in f(d_1)\} |$$

$$F_1 \in \mathcal{F}_1 \wedge f : F_1 \to \mathcal{F}_2\}.$$

LEMMA 1.5: For all $\mathcal{F}_i$, $(\mathcal{F}_1 \otimes \mathcal{F}_2) \otimes \mathcal{F}_3 \cong \mathcal{F}_1 \otimes (\mathcal{F}_2 \otimes \mathcal{F}_3)$.

The element of $\mathcal{F}_1 \otimes \mathcal{F}_2$ which is obtained from $F_1 \in \mathcal{F}_1$ and $f : F_1 \to \mathcal{F}_2$ is denoted by $F_1{}^f$.

LEMMA 1.6: Let $< \mathcal{L}_i, \mathcal{R}_i >$ be a weave of D_i for $i = 1, 2$. Define $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$ and $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2$. Then $< \mathcal{L}, \mathcal{R} >$ is a weave of $D = D_1 \times D_2$.

PROOF: Since $\mathcal{L}_1 \times \mathcal{L}_2 \subseteq \mathcal{L}$ and $\mathcal{R}_1 \times \mathcal{R}_2 \subseteq \mathcal{R}$, it suffices to show that, for every $L \in \mathcal{L}$ and every $R \in \mathcal{R}$, $\overline{\overline{L \cap R}} = 1$.

Let L be obtained from $L_1 \in \mathcal{L}_1$ and $f_1 : L_1 \to \mathcal{L}_2$, and let R be obtained from $R_1 \in \mathcal{R}_1$ and $g_1 : R_1 \to \mathcal{R}_2$. Let $L_1 \cap R_1 = \{d_1\}$, $f_1(d_1) = L_2$, and $g_1(d_1) = R_2$. Then the lemma is proved, since $L \cap R = \{d_1\} \times (L_2 \cap R_2)$. Q.E.D.

DEFINITION: The weave $< \mathcal{L}_1 \otimes \mathcal{L}_2, \mathcal{R}_1 \otimes \mathcal{R}_2 >$ is called a <u>tensor product</u> of $< \mathcal{L}_1, \mathcal{R}_1 >$ and $< \mathcal{L}_2, \mathcal{R}_2 >$.

Let $< \mathcal{L}_i, \mathcal{R}_i >$ be a normal weave of D_i, for each $i < \omega$. Define a game Γ as follows.

Let X be a subset of $\prod_i D_i$.

Stage 0: Players I and II <u>simultaneously</u> choose sets $L_0 \in \mathcal{L}_0$ and $R_0 \in \mathcal{R}_0$ respectively.

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Stage i: Players I and II simultaneously choose sets $L_i \in \mathcal{L}_i$ and $R_i \in \mathcal{R}_i$ respectively.

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For each i, let $L_i \cap R_i = \{d_i\}$. We say that player I wins the game iff the sequence $\langle d_0, d_1, d_2, \ldots \rangle$ is in X. Otherwise, player II wins. This is called a weave game with underlying set X.

Remark: This weave game is a generalization of the Gale-Stewart game. A Gale-Stewart game can be thought of as a weave game using only the trivial weave $\mathcal{L}_i = \{\{d\} | d \in D_i\}$, $\mathcal{R}_i = \{D_i\}$ for stages i, where player I is expected to choose, and $\mathcal{L}_j = \{D_j\}$, $\mathcal{R}_j = \{\{d\} | d \in D_j\}$ for stages j, where player II is expected choose.

Let $\tilde{\mathcal{L}} = \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \ldots$, $\tilde{\mathcal{R}} = \mathcal{R}_0 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \ldots$, and $\tilde{D} = D_0 \times D_1 \times D_2 \times \ldots$.

Now we translate "tensor product" into game terminology. A play of the game Γ is an element of $\tilde{D}$. A finite play of the game Γ is an initial segment of some play. A strategy for player I (II) in the game Γ is a function σ (whose domain is the set of finite plays) such that, for all $\langle d_0, \ldots, d_n \rangle$, $\sigma(\langle d_0,\ldots,d_n \rangle) \in \mathcal{L}_{n+1}$ $(\mathcal{R}_{n+1})$. (We will use these terms in connection with Gale-Stewart games also; see [3] for definitions.) Given a strategy σ, let $\hat{\sigma}$ be the set of all plays of Γ that can result from player I's using the strategy σ throughout the game.

Notice that $\tilde{\mathcal{L}} = \{\hat{\sigma} | \sigma$ is a strategy for I in $\Gamma\}$. This is because every set $\tilde{L}$ in $\tilde{\mathcal{L}}$ is of the form $L_0^{<f_1, f_2, \ldots>}$ where $L_0 \in \mathcal{L}_0$, $f_1 : L_0 \to \mathcal{L}_1$, $f_2 : L_0^{f_1} \to \mathcal{L}_2$, etc. So $\tilde{L} = \hat{\sigma}$, where σ is a strategy satisfying

$$\sigma(\Lambda) = L_0$$

$$\sigma(<d_0>) = f_1(d_0) \qquad \forall\, d_0 \in L_0$$

$$\vdots$$

$$\sigma(<d_0, \ldots, d_i>) = f_{i+1}(<d_0, \ldots, d_i>) \qquad \forall <d_0, \ldots, d_i> \in L_0^{<f_1, \ldots, f_i>}.$$

Conversely, given any strategy σ for I, define L_0, f_1, ..., f_{i+1}, ... according to the above equations. Then $\hat{\sigma} = L_0^{<f_1, f_2, \ldots>}$.

Similarly, $\tilde{\mathcal{R}} = \{\hat{\tau} | \tau$ is a strategy for II in $\Gamma\}$. (This is why we chose to make Γ a simultaneous-play game rather than a Gale-Stewart game. If we'd made Γ a Gale-Stewart game, this last claim would not have been true.)

We say that a game Γ with underlying set X is <u>determined</u> iff one of the players has a winning strategy. This is equivalent to

$$\exists\, \sigma(\text{strategy for I}) \;\; \exists\, \tau(\text{strategy for II}) \text{ s.t. } \hat{\sigma} \subseteq X \;\; \vee \;\; \hat{\tau} \subseteq \tilde{D} - X.$$

This in turn is equivalent to

$$\exists \tilde{L} \in \tilde{\mathcal{L}} \quad \exists \tilde{R} \in \tilde{\mathcal{R}} \quad \text{s.t. } \tilde{L} \subseteq X \vee \tilde{R} \subseteq \tilde{D} - X.$$

Therefore $< \tilde{\mathcal{L}}, \tilde{\mathcal{R}} >$ is $\mathcal{F}$-normal, for some particular family of sets $\mathcal{F}$, iff, for every X in $\mathcal{F}$, the game $< \mathcal{L}, \mathcal{R} >$ with underlying set X is determined.

Next we reduce the problem further to that of determinateness of Gale-Stewart games. We consider a weave game Γ and create from it an "equivalent" Gale-Stewart game Γ^*.

The game Γ^*: At stage i, player I chooses a set L_i in $\mathcal{L}_i$, <u>and then</u> player II chooses a set R_i in $\mathcal{R}_i$. Again, an element $<d_0, d_1, \ldots>$ of $\tilde{D}$ is formed, and player I wins iff $<d_0, d_1, \ldots> \in X$.

LEMMA 1.7: If player I has a winning strategy in Γ^*, then she has a winning strategy in Γ.

PROOF: Player I should play Γ using essentially the same strategy that she uses to win Γ^*.

Let σ^* be a winning strategy for I in Γ^*.

Let $<d_0, \ldots, d_n>$ be a finite play of Γ. Choose L_0, R_0, L_1, R_1, ..., L_n, R_n such that $L_i \cap R_i = \{d_i\}$ for all i = 1, ..., n. Define $\sigma(<d_0, \ldots, d_n>) = \sigma^*(<L_0, R_0, \ldots, L_n, R_n>)$. (Notice that a "finite play" of Γ^* is an initial segment of an element in $\mathcal{L}_0 \times \mathcal{R}_0 \times \mathcal{L}_1 \times \mathcal{R}_1 \times \ldots$. This is the only "difficulty" in applying a strategy for I in Γ^* to the game Γ.)

Then σ is a winning strategy for I in Γ. Q.E.D.

LEMMA 1.8: If player II has a winning strategy in Γ^*, then she has a winning strategy in Γ.

PROOF: (Intuitive Idea:) Games Γ and Γ^* are essentially the same except that player II seems to have an extra advantage in Γ^* that she doesn't have in Γ - that of knowing, at stage i, what player I's move for stage i will be before having to choose her own move for stage i. This turns out not to be an advantage at all (because of the normality of $<\mathcal{L}_i, \mathcal{R}_i>$). For instance, at stage 0, we let D_0^{II} be the set of all d in D_0 that players I and II can form (jointly) as part of τ^* - a winning strategy for II in Γ^*. It turns out that $\exists\, R \in \mathcal{R}_0$ s.t. $R \subseteq D_0^{II}$. In playing the game Γ, player II can simply choose R at stage 0, without knowing what player I's choice for stage 0 will be.

(Details:) Let f be the function that maps the plays of Γ^* onto the plays of Γ in the expected way (i.e. $f(<L_0,R_0,L_1,R_1,\ldots>) = <d_0,d_1,\ldots>$, where $L_i \cap R_i = \{d_i\}$ for all $i < \omega$.) Let τ^* be a winning strategy for II in Γ^*.

The set $\hat{\tau}^*$ is a II-imposed subgame (see [2]) of Γ^* in which all plays are wins for player II. Therefore, $f(\hat{\tau}^*)$ is a "subgame" of Γ in which all plays are wins for player II. We will define a strategy τ for II in Γ in such a way that $\hat{\tau} \subseteq f(\hat{\tau}^*)$. (Note that since $f(\hat{\tau}^*)$ is a closed set, we only have to define τ so as to make any <u>finite play</u> of $\hat{\tau}$ extendible to an element of $f(\hat{\tau}^*)$.) Thus τ will be a winning strategy for II in Γ.

Let D_0^{II} be the set of 0th entries of elements of $f(\hat{\tau}^*)$. (I.e., $D_0^{II} = \{d_0 \mid (\exists d_1,d_2,\ldots)\,[<d_0,d_1,d_2,\ldots> \in f(\hat{\tau}^*)]\}$.)

Claim: $\exists\, R \in \mathcal{R}_0$ s.t. $R \subseteq D_0^{II}$.

If not, then by normality $\exists\, L \in \mathcal{L}_0$ s.t. $L \subseteq D_0 - D_0^{II}$. This would imply that there is a move of player I in Γ^* (that of choosing the set L) that forces the game out of $\hat{\tau}^*$, contradicting the fact that $\hat{\tau}^*$ is a II-imposed subgame of Γ^*.

So define $\tau(\Lambda) = R$.

Now assume that Γ is in its ith stage, that $\langle d_0,\ldots,d_{i-1}\rangle$ has already been played, and (hypothesis of induction) that $\langle d_0,\ldots,d_{i-1}\rangle$ is an initial segment of some element of $f(\hat{\tau}^*)$.

Let $D_i^{II}\langle d_0,\ldots,d_{i-1}\rangle$ be the set of ith entries of those elements of $f(\hat{\tau}^*)$ that have $\langle d_0,\ldots,d_{i-1}\rangle$ as an initial segment. (I.e., $D_i^{II}\langle d_0,\ldots,d_{i-1}\rangle =$ $\{d_i \mid (\exists\, d_{i+1},d_{i+2},\ldots)\,[\langle d_0,\ldots,d_{i-1},d_i,d_{i+1},d_{i+2},\ldots\rangle \in f(\hat{\tau}^*)]\}$.) As before $\exists\, R \in \mathcal{R}_i$ s.t. $R \subseteq D_i^{II}\langle d_0,\ldots,d_{i-1}\rangle$. Define $\tau(\langle d_0,\ldots,d_{i-1}\rangle)$ to be this R.

Since $\hat{\tau} \subseteq f(\hat{\tau}^*)$, τ is a winning strategy for player II in Γ. Q.E.D.

Collecting the results of Lemmas 1.7 and 1.8, we have the following.

LEMMA 1.9: If Γ^* is determined, then so is Γ.

It is shown in [4] and [5] that any Gale-Stewart game Γ^*, whose underlying set X is a Borel set (as a subset of $\tilde{D}$), is determined. We therefore have, by Lemma 1.9, that any weave game, whose underlying set is a Borel game, is determined. Translating this back into the language of tensor products, we get the following.

THEOREM 1.10: Let $<\mathcal{L}_i, \mathcal{R}_i>$ be a normal weave of D_i, for all $i < \omega$. Let $\tilde{\mathcal{L}} = \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \ldots$, let $\tilde{\mathcal{R}} = \mathcal{R}_0 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \ldots$, and let $\tilde{D} = D_0 \times D_1 \times D_2 \times \ldots$. Let X be a Borel subset of $\tilde{D}$. Then there exists a set L in $\tilde{\mathcal{L}}$ such that $\tilde{L} \subseteq X$, or there exists a set $\tilde{R}$ in $\tilde{\mathcal{R}}$ such that $\tilde{R} \subseteq \tilde{D} - X$.

SECTION 2: THE FUNDAMENTAL THEOREM OF NORMAL WEAVES

In Sections 2 and 3 we show that normal weaves can be represented as trees. In Sections 4 through 7 we will present some applications of this result and discuss other ideas concerning weaves versus trees.

Before we state the Fundamental Theorem we need some definitions.

DEFINITION: (1) $\mathcal{L} = \bigvee_{i \varepsilon I} \mathcal{L}_i \iff \mathcal{L} = \bigcup_{i \varepsilon I} \mathcal{L}_i$

(2) $\mathcal{L} = \dot{\bigvee}_{i \varepsilon I} \mathcal{L}_i \iff \mathcal{L} = \bigvee_{i \varepsilon I} \mathcal{L}_i$ and $\forall\, i,j \; \varepsilon\; I$ if $i \neq j$, then $\mathrm{Alph}(\mathcal{L}_i) \cap \mathrm{Alph}(\mathcal{L}_j) = \emptyset$. (<u>direct disjunction</u>)

(3) $\mathcal{L} = \bigwedge_{i \varepsilon I} \mathcal{L}_i \iff \mathcal{L} = \{L \mid L = \bigcup_{i \varepsilon I} L_i$, where $L_i \;\varepsilon\; \mathcal{L}_i$ for for each $i \;\varepsilon\; I\}$.

(4) $\mathcal{L} = \dot{\bigwedge}_{i \varepsilon I} \mathcal{L}_i \iff \mathcal{L} = \bigwedge_{i \varepsilon I} \mathcal{L}_i$ and $\forall\, i,j \;\varepsilon\; I$ if $i \neq j$, then $\mathrm{Alph}(\mathcal{L}_i) \cap \mathrm{Alph}(\mathcal{L}_j) = \emptyset$. (<u>direct conjunction</u>)

(Let a be a letter, L be a set, and $\mathcal{L}$ be a family. The expression $L \;\dot{\vee}\; \mathcal{L}$ is an abbreviation $\{L\} \;\dot{\vee}\; \mathcal{L}$, and $a \;\dot{\vee}\; \mathcal{L}$ is an abbreviation for $\{\{a\}\} \;\dot{\vee}\; \mathcal{L}$.)

LEMMA 2.1: (a) The set G is a gcs for $\dot{\bigvee} \mathcal{L}_i$ iff, for each i_0, $G \cap \mathrm{Alph}(\mathcal{L}_{i_0})$ is a gcs for $\mathcal{L}_{i_0}$.

(b) The set G is a gcs for $\dot{\bigwedge}\mathcal{L}_i$ iff there is an i_0 such that G is a gcs for $\mathcal{L}_{i_0}$.

PROOF: (a) $\Longrightarrow$: Assume that G is a gcs for $\dot{\bigvee}_{i \in I} \mathcal{L}_i$. Choose $i_0 \in I$. For each $L_{i_0} \in \mathcal{L}_{i_0}$ we have $\overline{\overline{G \cap L_{i_0}}} = 1$. So $\overline{\overline{G \cap \text{Alph}(\mathcal{L}_{i_0}) \cap L_{i_0}}} = 1$. So $G \cap \text{Alph}(\mathcal{L}_{i_0})$ is a gcs for $\mathcal{L}_{i_0}$.

$\Longleftarrow$: Assume that $G \cap \text{Alph}(\mathcal{L}_{i_0})$ is a gcs for $\mathcal{L}_{i_0}$ (for each $i_0 \in I$). Let $i_1 \in I$, and let $L_{i_1} \in \mathcal{L}_{i_1}$. Then $G \cap L_{i_1} = G \cap \text{Alph}(\mathcal{L}_{i_1}) \cap L_{i_1}$, so $\overline{\overline{G \cap L_{i_1}}} = 1$. Therefore G is a gcs for $\dot{\bigvee}\mathcal{L}_i$.

(b) $\Longrightarrow$: Assume that G is a gcs for $\dot{\bigwedge}\mathcal{L}_i$. Then there is a unique $i_0 \in I$ such that $G \cap \text{Alph}(\mathcal{L}_{i_0}) \neq \emptyset$. (To show uniqueness let $G \cap \text{Alph}(\mathcal{L}_{i_0}) = a \in L_{i_0}$, and let $G \cap \text{Alph}(\mathcal{L}_{i_0'}) = a' \in L_{i_0'}$, where $i_0 \neq i_0'$. There is a set L in $\dot{\bigwedge}\mathcal{L}_i$ such that $L_{i_0}, L_{i_0'} \subseteq L$. Then $a, a' \in G \cap L$, which contradicts the fact that G is a gcs for $\dot{\bigwedge}\mathcal{L}_i$.)

Let $L_{i_0} \in \mathcal{L}_{i_0}$. Find a set L in $\dot{\bigwedge}\mathcal{L}_i$ such that $L_{i_0} \subseteq L$. Since $\overline{\overline{G \cap L}} = 1$, we must have that $\overline{\overline{G \cap L_{i_0}}} = 1$. Thus G is a gcs for $\mathcal{L}_{i_0}$.

$\Longleftarrow$: Assume that G is a gcs for $\mathcal{L}_{i_0}$. (Notice that this implies, by definition of "gcs", that $G \cap \text{Alph}(\mathcal{L}_{i_1}) = \emptyset$, for all other i_1 in I.) Let

$L \in \dot{\bigwedge}_{i \in I} \mathcal{L}_i$. Then $L = L_{i_0} \cup \bigcup_{\substack{i_1 \in I \\ i_1 \neq i_0}} L_{i_1}$. Also $\overline{\overline{G \cap L_{i_0}}} = 1$, and

$G \cap \bigcup L_{i_1} = \emptyset$. So $\overline{\overline{G \cap L}} = 1$. Thus G is a gcs for $\dot{\bigwedge} \mathcal{L}_i$. Q.E.D.

LEMMA 2.2: (The conjunctions and disjunctions in this lemma are non-trivial--that is, when we write, for instance, $\mathcal{L} = \dot{\bigvee}_{i \in I} \mathcal{L}_i$, we assume that $\overline{\overline{I}} > 1$ and $\mathcal{L}_i \neq \emptyset$ for each i.)

(a) Let $\mathcal{L} = \dot{\bigvee}_{i \in I} \mathcal{L}_i$. Then there is no collection $\{\mathcal{L}_j'\}_{j \in J}$ such that $\mathcal{L} = \dot{\bigwedge}_{j \in J} \mathcal{L}_j'$.

(b) Let $\mathcal{L} = \dot{\bigwedge}_{j \in J} \mathcal{L}_j$. Then there is no collection $\{\mathcal{L}_i'\}_{i \in I}$ such that $\mathcal{L} = \dot{\bigvee}_{i \in I} \mathcal{L}_i'$.

PROOF: Assume that $\mathcal{L} = \dot{\bigvee}_{i \in I} \mathcal{L}_i$ and $\mathcal{L} = \dot{\bigwedge}_{j \in J} \mathcal{L}_j'$, and that both of these expressions are non-trivial.

Choose an i in I, and let $\mathcal{L}_i$ be $\mathcal{L}_1$. Then $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$. Choose a j in J, and let $\mathcal{L}_j'$ be $\mathcal{L}_3$. Then $\mathcal{L} = \mathcal{L}_3 \dot{\wedge} \mathcal{L}_4$. Now $\text{Alph}(\mathcal{L}_1) \cap \text{Alph}(\mathcal{L}_3)$ is non-empty. (Let $L \in \mathcal{L}_1$. Then $L \in \mathcal{L}$. So $L = L_3 \cup L_4$, where $L_3 \in \mathcal{L}_3$ and $L_4 \in \mathcal{L}_4$. Then $L_3 \subseteq \text{Alph}(\mathcal{L}_1) \cap \text{Alph}(\mathcal{L}_3)$.) Let $a \in \text{Alph}(\mathcal{L}_1) \cap \text{Alph}(\mathcal{L}_3)$. Similarly let $b \in \text{Alph}(\mathcal{L}_2) \cap \text{Alph}(\mathcal{L}_4)$.

Since $a \in \text{Alph}(\mathcal{L}_1)$, $b \in \text{Alph}(\mathcal{L}_2)$ and $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$, there is no set L in $\mathcal{L}$ such that $a, b \in \mathcal{L}$. But since $a \in \text{Alph}(\mathcal{L}_3)$ and $b \in \text{Alph}(\mathcal{L}_4)$, there $\underline{\text{is}}$ a set L in $\mathcal{L}$ such that a, b ε $\mathcal{L}$ (because we can let a ε L_3 and b ε L_4; then a, b ε $L_3 \cup L_4 \varepsilon \mathcal{L}$.) This is a contradiction. Q.E.D.

LEMMA 2.3: (a) If $\mathcal{L} = \dot{\bigvee} \mathcal{L}_i$, then $\mathcal{G}(\mathcal{L}) = \dot{\bigwedge} \mathcal{G}(\mathcal{L}_i)$.

(b) If $\mathcal{L} = \dot{\bigwedge} \mathcal{L}_i$, then $\mathcal{G}(\mathcal{L}) = \dot{\bigvee} \mathcal{G}(\mathcal{L}_i)$.

PROOF: This is a corollary of Lemma 2.1.

(a) Let $\mathcal{L} = \dot{\bigvee} \mathcal{L}_i$. A set G is a gcs for $\mathcal{L}$ iff, for each i_0, $G \cap \text{Alph}(\mathcal{L}_{i_0})$ is a gcs for $\mathcal{L}_i$. This happens iff $G = \bigcup_{i \varepsilon I} G \cap \text{Alph}(\mathcal{L}_i)$, where $G \cap \text{Alph}(\mathcal{L}_i) \varepsilon \mathcal{G}(\mathcal{L}_i)$, for each i ε I. This, in turn, is true iff G ε $\dot{\bigwedge}_{i \varepsilon I} \mathcal{G}(\mathcal{L}_i)$.

(b) The proof of (b) is similar. Q.E.D.

The object now is to prove the following.

THE FUNDAMENTAL THEOREM OF NORMAL WEAVES: Let $\mathcal{L}$ be a normal weave. Then either $\mathcal{L} = \dot{\bigwedge} \mathcal{L}_i$ or $\mathcal{L} = \dot{\bigvee} \mathcal{L}_i$, where, in either case, $\mathcal{L}_i$ is normal for each i (and the conjunction or disjunction is non-trivial).

The Fundamental Theorem was first proved by Gaisi Takeuti for the case where the alphabet of $\mathcal{L}$ is finite. It was proved independently, and the new proof was expanded to cover the infinite case, as part of the research for this thesis.

<u>Outline and Motivation for the Proof</u>: Every weave is naturally a disjunction--it's a disjunction of its sets:

$$\mathcal{L} = L_0 \vee L_1 \vee \ldots \vee L_5$$

where $L_0, L_1, \ldots, L_5$ are the sets in $\mathcal{L}$. The problem is that this disjunction may not be direct. For instance, the letter a may appear in both L_0 and L_1. In this case we begin by "factoring out" a to get

$$a \mathbin{\dot{\wedge}} (L_0' \vee L_1') \vee L_2 \vee \ldots \vee L_5 .$$

In fact, we can find a gcs {a,b,c} and factor it out:

$$a \mathbin{\dot{\wedge}} (L_0' \vee L_1') \vee b \mathbin{\dot{\wedge}} (L_2' \vee L_3') \vee c \mathbin{\dot{\wedge}} (L_4' \vee L_5').$$

Now we would be done if the outermost disjunctions were all direct. Unfortunately this isn't always the case. The families $L_0' \vee L_1'$ and $L_2' \vee L_3'$ may have letters in common. We must show that, in such a case, the letters that these two families have in common can be factored out, and that they can be factored out of both families in the same way.

$$\left[a \mathbin{\dot{\wedge}} (L_0'' \vee L_1'') \mathbin{\dot{\vee}} b \mathbin{\dot{\wedge}} (L_2'' \vee L_3'') \right] \mathbin{\dot{\wedge}} \mathcal{L}^* \vee c \mathbin{\dot{\wedge}} (L_4' \vee L_5').$$

Still the outermost disjunction may not be direct. We need an extra lemma to show that sets of letters that $L_0'' \vee L_1''$ and $L_4' \vee L_5'$ have in common can be factored out (Lemma 2.9).

Finally we must show that this process of factoring stops after finitely many steps.

Now we begin the proof of the theorem.

LEMMA 2.4: Let $S, T \subseteq \mathrm{Alph}(\mathcal{L})$, $S \cap T = \emptyset$, and $S, T \neq \emptyset$. Assume that, for each G which is a gcs of $\mathcal{L}$, either $G \subseteq S$ or $G \subseteq T$. Then $\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{L}_S \dot{\wedge} \mathcal{L}_T$, where $S = \mathrm{Alph}(\mathcal{L}_S)$ and $T = \mathrm{Alph}(\mathcal{L}_T)$.

PROOF: Let $\mathcal{R}_S$ be all gcs's of $\mathcal{L}$ that are subsets of S, and let $\mathcal{R}_T$ be all gcs's of $\mathcal{L}$ that are subsets of T. Let $\mathcal{R} = \mathcal{R}_S \dot{\vee} \mathcal{R}_T$. Then $\mathcal{R}$ is the family consisting of all gcs's of $\mathcal{L}$. By Lemma 2.3, $\mathcal{G}(\mathcal{R}) = \mathcal{G}(\mathcal{R}_S) \dot{\wedge} \mathcal{G}(\mathcal{R}_T)$. The family $\mathcal{G}(\mathcal{R})$ is equal to $\mathcal{G}\mathcal{G}(\mathcal{L})$. Let $\mathcal{L}_S$ be $\mathcal{G}(\mathcal{R}_S)$, and let $\mathcal{L}_T$ be $\mathcal{G}(\mathcal{R}_T)$. Then we have $\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{L}_S \dot{\wedge} \mathcal{L}_T$. Q.E.D.

COROLLARY: If, in addition to the hypothises of Lemma 2.4, we have that $\mathcal{L}$ is normal, then $\mathcal{L} = \mathcal{L}_S \dot{\wedge} \mathcal{L}_T$.

PROOF: If $\mathcal{L}$ is normal, then Lemma 1.2 tells us that $\mathcal{L} = \mathcal{G}\mathcal{G}(\mathcal{L})$. Q.E.D.

For Lemmas 2.5 through 2.8, let $\mathcal{L}$ be normal and let $\mathcal{L} = \bigvee_{i \in I} (a_i \dot{\wedge} \mathcal{L}_i)$, where $\{a_i\}_{i \in I}$ is a gcs for $\mathcal{L}$ and, for each $i \neq j$, $a_i \neq a_j$.

LEMMA 2.5: (1) For any $J \subseteq I$, $\bigvee_{i \in j} (a_i \dot{\wedge} \mathcal{L}_i)$ is normal.

(2) For any $i \in I$, $a_i \dot{\wedge} \mathcal{L}_i$ is normal.

(3) For any $i \in I$, $\mathcal{L}_i$ is normal.

PROOF: (1) Assume that G is a gcs of $\bigvee_{i \in J} (a_i \dot{\wedge} \mathcal{L}_i)$. Consider $G' = G \cup \{a_i | i \notin J\}$. The set G' is a cs for $\mathcal{L}$. Since $\mathcal{L}$ is normal, there is a set $G'' \subseteq G'$ such that G'' is a gcs for $\mathcal{L}$. The set $G'' - \{a_i | i \notin J\}$ is a subset of G. It is also a cs for $\bigvee_{i \in J} (a_i \dot{\wedge} \mathcal{L}_i)$, since $\{a_i | i \notin J\}$ and $\text{Alph}(\bigvee_{i \in J} a_i \dot{\wedge} \mathcal{L}_i)$ have no letters in common. Also, since $\overline{\overline{G'' \cap L}} \leq 1$ for each set L in $\mathcal{L}$, $\overline{\overline{(G'' - \{a_i | i \notin J\}) \cap L}} \leq 1$ for each set L in $\bigvee_{i \in J} (a_i \dot{\wedge} \mathcal{L}_i)$. So $G'' - \{a_i | i \notin J\}$ is a gcs for $\bigvee_{i \in J} (a_i \dot{\wedge} \mathcal{L}_i)$. Thus $\bigvee_{i \in J} (a_i \dot{\wedge} \mathcal{L}_i)$ is normal.

(2) This is a corollary of (1).

(3) Let C be a cs for $\mathcal{L}_i$. Then C is a cs for $a_i \dot{\wedge} \mathcal{L}_i$. So, by (2), there is a set $G \subseteq C$ such that G is a gcs for $a_i \dot{\wedge} \mathcal{L}_i$. But then $a_i \notin G$, so (by Lemma 2.1), G is a gcs for $\mathcal{L}_i$. Q.E.D.

LEMMA 2.6: Assume that Alph($\mathcal{L}_0$) $\cap$ Alph($\mathcal{L}_1$) = C $\neq \emptyset$. If G is a gcs of $\mathcal{L}_0$ and G $\cap$ C $\neq \emptyset$, then G $\cap$ C is a gcs of $\mathcal{L}_1$. (Likewise, if G is a gcs of $\mathcal{L}_1$ and G $\cap$ C $\neq \emptyset$, then G $\cap$ C is a gcs of $\mathcal{L}_0$.).

PROOF: Let G be a gcs of $\mathcal{L}_0$, and assume that G $\cap$ C $\neq \emptyset$. Consider G $\cup \{a_1\}$. This is a cs of $a_0 \dot{\wedge} \mathcal{L}_0 \vee a_1 \dot{\wedge} \mathcal{L}_1$, which is normal. So there is a a gcs G' of $a_0 \dot{\wedge} \mathcal{L}_0 \vee a_1 \dot{\wedge} \mathcal{L}_1$ satisfying G' $\subseteq$ G $\cup$ $\{a_1\}$. Since $a_1 \notin a_0 \dot{\wedge} \mathcal{L}_0$, we must have that G' - $\{a_1\}$ is a gcs for $a_0 \dot{\wedge} \mathcal{L}_0$. But G' - $\{a_1\}$ $\subseteq$ G, and G is also a gcs for $a_0 \dot{\wedge} \mathcal{L}_0$. So G' - $\{a_1\}$ = G (by Lemma 1.1).

So either G' = G $\cup$ $\{a_1\}$ or G' = G. The set G $\cup$ $\{a_1\}$ cannot be a gcs for $a_0 \dot{\wedge} \mathcal{L}_0 \vee a_1 \dot{\wedge} \mathcal{L}_1$, since it contains <u>both</u> a_1 <u>and</u> letters in G $\cap$ C. ($\exists$ L ε $a_1 \dot{\wedge} \mathcal{L}_1$ $\exists$c ε G $\cap$ C s.t. a_1, c ε L.) So G' = G.

Thus G is a gcs for $a_0 \dot{\wedge} \mathcal{L}_0 \vee a_1 \dot{\wedge} \mathcal{L}_1$. So G meets every set in $a_1 \dot{\wedge} \mathcal{L}_1$ in one and only one place, and $a_1 \notin$ G. So G meets every set in $\mathcal{L}_1$ in one and only one place. So G $\cap$ Alph($\mathcal{L}_1$) is a gcs for $\mathcal{L}_1$. But G $\cap$ Alph($\mathcal{L}_1$) = G $\cap$ C. Therefore G $\cap$ C is a gcs for $\mathcal{L}_1$. Q.E.D.

LEMMA 2.7: Assume that Alph($\mathcal{L}_0$) $\cap$ Alph($\mathcal{L}_1$) = C $\neq \emptyset$. Then, for each G which is a gcs of $\mathcal{L}_0$, either G $\subseteq$ C or G $\subseteq$ Alph($\mathcal{L}_0$) - C. Similarly, for each G which is a gcs of $\mathcal{L}_1$, either G $\subseteq$ C or G $\subseteq$ Alph($\mathcal{L}_1$) - C.

PROOF: Let G be a gcs of $\mathcal{L}_0$ and assume that $G \cap C \neq \emptyset$. Then by Lemma 2.6, $G \cap C$ is a gcs of $\mathcal{L}_1$. By applying Lemma 2.6 to $G \cap C$, we get that $(G \cap C) \cap C$ is a gcs of $\mathcal{L}_0$. So $G \cap C$ and G are both gcs's of $\mathcal{L}_0$. So, by Lemma 1.1, $G \cap C = G$. So $G \subseteq C$. Q.E.D.

LEMMA 2.8: Assume that $\text{Alph}(\mathcal{L}_0) \cap \text{Alph}(\mathcal{L}_1) = C \neq \emptyset$. Then $\mathcal{L}_0 = \mathcal{L}_0' \dot{\wedge} \mathcal{L}_C$ and $\mathcal{L}_1 = \mathcal{L}_1' \dot{\wedge} \mathcal{L}_C$, where $C = \text{Alph}(\mathcal{L}_C)$.

PROOF: This follows by Lemma 2.4 and Lemma 2.7. Q.E.D.

LEMMA 2.9: Let $\mathcal{L} = [(\mathcal{L}_1 \vee \mathcal{L}_2) \dot{\wedge} \mathcal{L}_3] \vee \mathcal{L}_4$, where $\mathcal{L}$ is normal and $\mathcal{L}_3$ is not a conjunct of $\mathcal{L}_4$. Let $\text{Alph}(\mathcal{L}_1) \cap \text{Alph}(\mathcal{L}_4) = C \neq \emptyset$. Then $C \subseteq \text{Alph}(\mathcal{L}_2)$. Also, $\mathcal{L}_1 \vee \mathcal{L}_2 = \mathcal{L}' \dot{\wedge} \mathcal{L}_C$ and $\mathcal{L}_4 = \mathcal{L}_4' \dot{\wedge} \mathcal{L}_C$, where $C = \text{Alph}(\mathcal{L}_C)$.

PROOF: We will show that $C \subseteq \text{Alph}(\mathcal{L}_2)$. The rest of the proof is similar to the proofs of Lemma 2.6 through 2.8.

The family $\mathcal{L}$ can be written in the form

$$(\mathcal{L}_1 \dot{\wedge} \mathcal{L}_3) \vee (\mathcal{L}_2 \dot{\wedge} \mathcal{L}_3) \vee \mathcal{L}_4.$$

Let G_4 be a gcs for $\mathcal{L}_4$ satisfying $G_4 \cap C \neq \emptyset$. Then as we have shown in the proof of Lemma 2.6, G_4 is a gcs for $\mathcal{L}_1$.

Let G_3 be a gcs for $\mathcal{L}_3$. The set $G_3 \cup G_4$ is a cs for $\mathcal{L}$. Now $G_3 \subseteq \mathrm{Alph}(\mathcal{L}_3)$, $G_4 \subseteq \mathrm{Alph}(\mathcal{L}_1)$, and $\mathcal{L}_1 \dot{\wedge} \mathcal{L}_3$ is part of our expression for $\mathcal{L}$, so, in order to obtain a gcs for $\mathcal{L}$, we must eliminate (from $G_3 \cup G_4$) either all letters in G_3 or all letters in G_4.

Case (1) We can always eliminate G_4. Then every gcs for $\mathcal{L}_3$ is a gcs for $\mathcal{L}_4$. Let $S = \mathrm{Alph}(\mathcal{L}_3)$ and $T = \mathrm{Alph}(\mathcal{L}_4) - \mathrm{Alph}(\mathcal{L}_3)$, and apply the Corollary to Lemma 2.4.This gives us the fact $\mathcal{L}_3$ is a conjunct of $\mathcal{L}_4$, contradicting the hypothesis.

Case (2) We cannot always eliminate G_4. Then there is a set G_3' which is a gcs for $\mathcal{L}_3$ and is not a gcs for $\mathcal{L}$.

Let G_4' be <u>any</u> gcs for $\mathcal{L}_4$ satisfying $G_4' \cap C \neq \emptyset$. Then $G_3' \cup G_4'$ is a cs for $\mathcal{L}$, and G_3' is not a gcs for $\mathcal{L}$, so, since we have to eliminate G_3' or G_4', G_4' is a gcs for $\mathcal{L}$.

Thus any set G_4', which satisfies $G_4' \cap C \neq \emptyset$ and is a gcs for $\mathcal{L}_4$, is also a gcs for $\mathcal{L}$. So G_4' must be a gcs for $\mathcal{L}_2 \dot{\wedge} \mathcal{L}_3$. But $C \cap \mathrm{Alph}(\mathcal{L}_3) = \emptyset$. So G_4' is a gcs for $\mathcal{L}_2$.

We have shown that <u>any</u> such set G_4' is a gcs for $\mathcal{L}_2$. Thus $C \subseteq \mathrm{Alph}(\mathcal{L}_2)$. Q.E.D.

LEMMA 2.10: Let $\mathcal{L}$ be of the form

$$[\ldots((((\mathcal{L}_0 \mathbin{\dot{\wedge}} \mathcal{M}_0) \mathbin{\dot{\vee}} \mathcal{L}_1) \mathbin{\dot{\wedge}} \mathcal{M}_1) \mathbin{\dot{\vee}} \mathcal{L}_2) \mathbin{\dot{\wedge}} \ldots].$$

Then $\mathcal{L}$ is not normal.

PROOF: If $\mathcal{L}$ is of that form, then $\mathcal{L}$ can also be written as

$$(\mathcal{L}_0 \mathbin{\dot{\wedge}} \dot{\bigwedge_{0 \leq i < \omega}} \mathcal{M}_i) \vee (\mathcal{L}_1 \mathbin{\dot{\wedge}} \dot{\bigwedge_{1 \leq i < \omega}} \mathcal{M}_i) \vee$$

$$(\mathcal{L}_2 \mathbin{\dot{\wedge}} \dot{\bigwedge_{2 \leq i < \omega}} \mathcal{M}_i) \vee \cdots .$$

For each i satisfying $0 \leq i < \omega$, let C_i be a gcs for $\mathcal{M}_i$, and let C be $\bigcup_{0 \leq i < \omega} C_i$. The set C is a cs for $\mathcal{L}$.

Assume that $G \subseteq C$, and G is a gcs for $\mathcal{L}$. Then there is a <u>unique</u> i_0 satisfying $G \cap C_{i_0} \neq \emptyset$. (Otherwise we have $c_{i_0} \in C_{i_0} \cap G$ and $c_{i_1} \in C_{i_1} \cap G$. Let $M_0 \cup M_1 \cup \ldots \cup M_{i_0} \cup \ldots M_{i_1} \cup \ldots = M \in \dot{\bigwedge_{0<i<\omega}} \mathcal{M}_i$, where $c_{i_0} \in M_{i_0}$ and $c_{i_1} \in M_{i_1}$. Then $c_{i_0}, c_{i_1} \in G$ <u>and</u> $c_{i_0}, c_{i_1} \in M$, so G is not a gcs for $\mathcal{L}$.)

Thus by Lemma 1.1, $G = C_{i_0}$. But $\mathrm{Alph}(\mathcal{M}_{i_0}) \cap \mathrm{Alph}(\mathcal{L}_{i_0+1} \mathbin{\dot{\wedge}} \dot{\bigwedge_{i_0+1 \leq i<\omega}} \mathcal{M}_i) = \emptyset$, so C_{i_0} (and thus G) cannot be a cs for $\mathcal{L}_{i_0+1} \dot{\wedge} \dot{\bigwedge}_{i_0+1\leq i<\omega} \mathcal{M}_i$, contradicting the fact that it is a gcs for $\mathcal{L}$. Therefore, $\mathcal{L}$ is not normal. Q.E.D.

PROOF OF THE FUNDAMENTAL THEOREM: We give a procedure for working $\mathcal{L}$ into the form of a direct conjunction or a direct disjunction.

(I) Begin with $\mathcal{L}$ in the form

$$L_0 \vee L_1 \vee L_2 \vee \dots \vee L_\alpha \vee \dots,$$

where the L_i's are the sets in $\mathcal{L}$.

(II) Choose a gcs $\{a_0, a_1, \dots\}$ for $\mathcal{L}$, and rearrange the L_i's so that

$$a_0 \in L_{0_0},\ a_0 \in L_{0_1},\ \dots$$

$$a_1 \in L_{1_0},\ a_1 \in L_{1_1},\ \dots$$

$$\vdots$$

So you have

$$(L_{0_0} \vee L_{0_1} \vee \dots) \vee (L_{1_0} \vee L_{1_1} \vee \dots) \vee \dots \vee$$

$$(L_{\beta_0} \vee L_{\beta_1} \vee \dots) \vee \dots,$$

or

$$(a_0 \dot{\wedge} \mathcal{L}_0) \vee (a_1 \dot{\wedge} \mathcal{L}_1) \vee \dots \vee (a_\beta \dot{\wedge} \mathcal{L}_\beta) \vee \dots, (*)$$

where $\mathcal{L}_n = \bigvee_k (L_{n_k} - \{a_n\})$.

(III) Case (1) The alphabets of the $\mathcal{L}_i$'s are pairwise disjoint. Then we are done.

Case (2) Without loss of generality assume that $\text{Alph}(\mathcal{L}_0) \cap \text{Alph}(\mathcal{L}_1) = C \neq \emptyset$. Rearrange the disjuncts in (*) so that $\mathcal{L}_0, \dots, \mathcal{L}_\alpha, \dots, \alpha < \gamma$, all have C in common; i.e., $\forall\, i \neq j$ $(i,j < \gamma)$, $\text{Alph}(\mathcal{L}_i) \cap \text{Alph}(\mathcal{L}_j) \supseteq C$. Then by Lemma 2.8 we have

$$\mathcal{L}_0 = \mathcal{L}_0' \dot{\wedge} \mathcal{L}_C$$

$$\mathcal{L}_1 = \mathcal{L}_1' \dot{\wedge} \mathcal{L}_C$$

$$\vdots$$

where $C = \text{Alph}(\mathcal{L}_C)$. So we have

$$a_0 \dot{\wedge} \mathcal{L}_0' \dot{\wedge} \mathcal{L}_C \vee a_1 \dot{\wedge} \mathcal{L}_1' \dot{\wedge} \mathcal{L}_C \vee \dots$$

$$a_\alpha \dot{\wedge} \mathcal{L}_\alpha' \dot{\wedge} \mathcal{L}_C \vee \dots \vee a_\gamma \dot{\wedge} \mathcal{L}_\gamma \vee \dots,$$

or

$$[(a_0 \dot{\wedge} \mathcal{L}_0') \vee (a_1 \dot{\wedge} \mathcal{L}_1') \vee \ldots \vee (a_\alpha \dot{\wedge} \mathcal{L}_\alpha') \vee \ldots]$$

$$\dot{\wedge} \mathcal{L}_C \vee (a_\gamma \dot{\wedge} \mathcal{L}_\gamma) \vee \ldots . \qquad (**)$$

(IV) Case(1) No $\mathcal{L}_\delta$, for any $\delta \geq \gamma$ has any letters in common with $\mathcal{L}_C$. Then the disjunction in (**) following $\mathcal{L}_C$ is direct because, by Lemma 2.9, no $\mathcal{L}_\delta$, for any $\delta \geq \gamma$, can have any letter in common with any $\mathcal{L}_\alpha'$, for any $\alpha < \gamma$. Go back to the beginning of step (III), this time working on the $\mathcal{L}_i$'s for $i \geq \gamma$.

Case(2) Without loss of generality assume that $\text{Alph}(\mathcal{L}_\gamma) \cap C = K \neq \emptyset$. Rearrange and renumber the disjuncts in (**) so that $\mathcal{L}_\gamma$, $\mathcal{L}_{\gamma+1}$, ..., $\mathcal{L}_\pi$, ..., $\pi < \sigma$, all have K in common; i.e., $\forall\ i \neq j\ (\beta \leq i,\ j \leq \sigma)$, $\text{Alph}(\mathcal{L}_i) \cap \text{Alph}(\mathcal{L}_j) \supseteq K$. Then by Lemma 2.8 we have

$$\mathcal{L}_C = \mathcal{L}_C' \dot{\wedge} \mathcal{L}_K$$

$$\mathcal{L}_\gamma = \mathcal{L}_\gamma' \dot{\wedge} \mathcal{L}_K$$

$$\vdots$$

$$\mathcal{L}_\pi = \mathcal{L}_\pi' \dot{\wedge} \mathcal{L}_K$$

$$\vdots$$

where $K = \text{Alph}(\mathcal{L}_K)$. So we have

$$[\cdots] \mathbin{\dot\wedge} \mathcal{L}_{C'} \mathbin{\dot\wedge} \mathcal{L}_K \vee a_\gamma \mathbin{\dot\wedge} \mathcal{L}_{\gamma'} \mathbin{\dot\wedge} \mathcal{L}_K \vee \cdots \vee$$

$$a_\pi \mathbin{\dot\wedge} \mathcal{L}_{\pi}' \mathbin{\dot\wedge} \mathcal{L}_K \vee \cdots \vee a_\sigma \mathbin{\dot\wedge} \mathcal{L}_\sigma \vee \cdots,$$

or

$$[[\cdots] \mathbin{\dot\wedge} \mathcal{L}_{C'} \vee (a_\gamma \mathbin{\dot\wedge} \mathcal{L}_{\gamma'}) \vee \cdots \vee (a_\pi \mathbin{\dot\wedge} \mathcal{L}_{\pi}') \vee \cdots]$$

$$\mathbin{\dot\wedge} \mathcal{L}_K \vee (a_\sigma \mathbin{\dot\wedge} \mathcal{L}_\sigma) \vee \cdots .$$

Now repeat step (IV) for $\mathcal{L}_\delta$'s such that $\delta \geq \sigma$.

We must show that we won't get stuck in an infinite loop of applications of step (IV) Case (2), because such a loop could lead to an expression of the form

$$[\cdots((((\mathcal{L}_0 \mathbin{\dot\wedge} \mathcal{L}_1) \mathbin{\dot\vee} \mathcal{L}_2) \mathbin{\dot\wedge} \mathcal{L}_3) \mathbin{\dot\vee} \mathcal{L}_4) \mathbin{\dot\wedge} \cdots],$$

which has no outermost connective. We will have more to say about this situation later on. For now suffice it to say that this is impossible by Lemma 2.10.

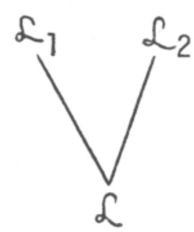
This completes the proof of the Fundamental Theorem.

SECTION 3: TREES

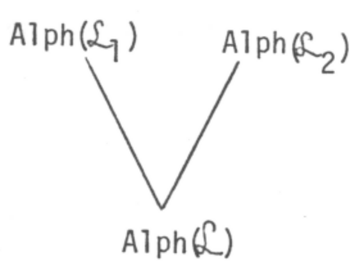
Before listing corollaries to The Fundamental Theorem we want to define "decomposable" weaves and show how these decomposable weaves correspond to trees. The Fundamental Theorem states essentially that every normal weave is decomposable. We will distinguish between two different types of decomposable weaves (stopped and continuing) and prove tree theorems for each.

The general idea of trees and decomposable weaves is highly intuitive, but the formal definitions and proofs are rather messy. We've tried to simplify the formalism, mainly by glossing over many steps (hopefully not at the expense of clarity). The reader should feel free to skip any definition, or the proof of any claim, that seems obvious, as there are no surprises in the formal treatment.

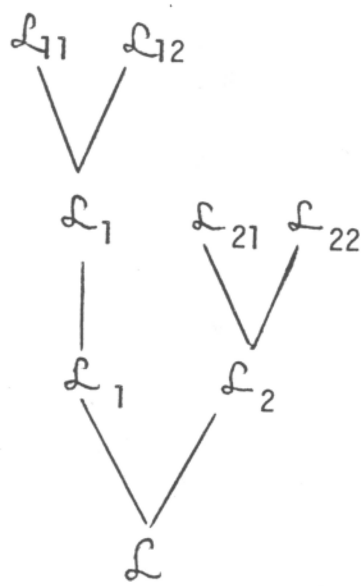
Let $< \mathcal{L}, \mathcal{R} >$ be a weave and let $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$. We can represent this in the following form.



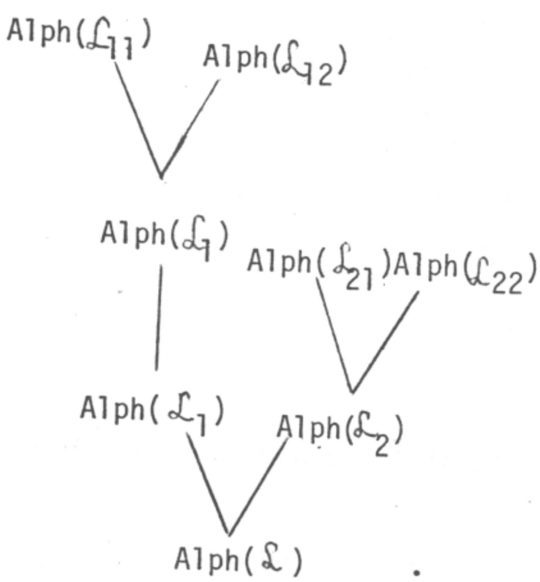
We can also represent it using the corresponding alphabets.



Now say that $\mathcal{L}_1 = \mathcal{L}_{11} \dot{\vee} \mathcal{L}_{12}$ and $\mathcal{L}_2 = \mathcal{L}_{21} \dot{\wedge} \mathcal{L}_{22}$. We can represent this by

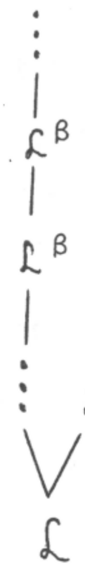


or by

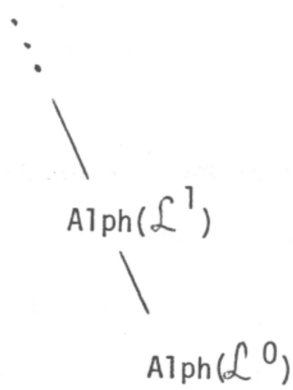


.

We continue on in the same fashion. If we come to a "component" $\mathcal{L}^{\beta}$ that cannot be decomposed (perhaps because it's a singleton set), we write



(or likewise with Alph($\mathcal{L}^{\beta}$)). If we have a branch of limit-ordinal length (ω, for instance)



then above the branch, on a node of level ω, we put $\bigcap_{i < \omega}$ Alph($\mathcal{L}^{i}$). (If $\bigcap_{i < \omega}$ Alph($\mathcal{L}^{i}$) $\neq \emptyset$, we put $\{L \cap \bigcap_{i < \omega}$ Alph($\mathcal{L}^{i}$) $\mid L \varepsilon \mathcal{L}\}$. If $\bigcap_{i < \omega}$ Alph($\mathcal{L}^{i}$) $= \emptyset$, we put Λ. In this respect the tree of alphabets is much less complicated notationally than the tree of weaves.) And so on.

We continue to do this until we reach a level γ on the tree where no "components" can be decomposed (i.e. all nodes become unbranching). We stop building the tree at this level and call these nodes of level γ the final nodes of the tree.

The weave $\mathcal{L}$ is said to be decomposable iff all final nodes are labeled with singleton sets or with Λ. From now on in Section 3 we consider only decomposable families $\mathcal{L}$. A branch b on the tree is called stopped iff its final node is labeled with a singleton set. Otherwise it is called continuing. If all branches on the tree for $\mathcal{L}$ are stopped, then $\mathcal{L}$ (and its corresponding tree) are said to be stopped. Otherwise they are said to be continuing.

We adopt, for trees, some of the standard terminology for games. (See [3].) Nodes that appear on even-numbered levels of the tree are nodes of even level or nodes of even parity. When it is convenient we will refer to these nodes as I-nodes, because we reserve these nodes as places where player I is to make a choice. Likewise, nodes that appear on odd-numbered levels of the tree are called nodes of odd level, nodes of odd parity, or II-nodes. A strategy for player I (a I-strategy) and a strategy for player II (II-strategy) are defined as in [3]. The set of I-strategies is denoted Σ_I and the set of II strategies is denoted Σ_{II}. If σ is a I-strategy, we let $\hat{\sigma}$ be the set of all stopped branches that σ passes through (or, equivalently, the set of all final nodes not labeled with "Λ" that σ passes through). If $\mathcal{T}$ is a tree, we let $\hat{\mathcal{T}}$ be the set of all branches of $\mathcal{T}$.

All of the trees that we deal with this section have final nodes. That is to say, we do not consider trees with branches of length λ, where λ is a limit ordinal. The purpose of this is to keep the exposition as simple as possible, and there is no loss of generality in doing it. For instance, instead of having a branch of length ω, we have a branch of length $\omega + 1$, where the ωth node gets the label "Λ".

Now we need a detour to discuss "subalphabets" and "components". This will make the proof of Theorem 3.1 easier.

Consider a statement in the propositional calculus. The usual way to define what we mean by a "subformula" of that statement is to define it inductively. But since our statements (our weaves) may take more than finitely many steps to decompose, we want a more direct way of referring to subformulas of a statement.

DEFINITION: Let $A \subseteq \mathrm{Alph}(\mathcal{L})$. Then $\mathcal{L} \restriction A$ is $\{L \cap A \mid L \in \mathcal{L} \wedge L \cap A \neq \emptyset\}$.

DEFINITION: Let $A \subseteq \mathrm{Alph}(\mathcal{L})$. The set A is a <u>subalphabet</u> of $\mathcal{L}$ iff

$$\forall L \in \mathcal{L} \quad \forall R \in \mathcal{G}(\mathcal{L}) \, [L \cap A \neq \emptyset \wedge R \cap A \neq \emptyset \Rightarrow L \cap R \subseteq A].$$

PROPOSITION 3.1: If A is a subalphabet of $\mathcal{L}$, then $\mathcal{L} \restriction A$ is a weave.

PROOF: Let $d \in A$. Take a set R in $\mathcal{G}(\mathcal{L})$ such that $d \in R$. The set $R \cap A$ is in $\mathcal{G}(\mathcal{L}) \restriction A$, and $d \in R \cap A$. So, for each d in $\mathrm{Alph}(\mathcal{L} \restriction A)$, there is a set R' in $\mathcal{G}(\mathcal{L}) \restriction A$ such that $d \in R'$.

We need only show that $R \cap A$ is a gcs for $\mathcal{L} \upharpoonright A$. Let $L \cap A \in \mathcal{L} \upharpoonright A$, and let $L \cap A \neq \emptyset$. Also let $R \cap A \in \mathcal{G}(\mathcal{L}) \upharpoonright A$, and let $R \cap A \neq \emptyset$. Then by definition of a subalphabet, $L \cap R \subseteq A$, so $(L \cap A) \cap (R \cap A) = (L \cap R) \cap A = L \cap R$. Since $\overline{\overline{L \cap R}} = 1$, we have that $\overline{\overline{(L \cap A) \cap (R \cap A)}} = 1$. Therefore $R \cap A$ is a gcs for $\mathcal{L} \upharpoonright A$. Q.E.D.

DEFINITION: Let A be a subalphabet of $\mathcal{L}$. Then $\mathcal{L} \upharpoonright A$ is called a <u>component</u> of $\mathcal{L}$.

PROPOSITION 3.2: If A is a subalphabet of $\mathcal{L}$ and B is a subalphabet of $\mathcal{L} \upharpoonright A$, then B is a subalphabet of $\mathcal{L}$.

PROOF: Let $L \in \mathcal{L}$ and $R \in \mathcal{G}(\mathcal{L})$ and assume that $L \cap B \neq \emptyset$ and that $R \cap B \neq \emptyset$. We must show that $L \cap R \subseteq B$.

The set B is a subset of A, so $L \cap A \supseteq L \cap B \neq \emptyset$ and $R \cap A \supseteq R \cap B \neq \emptyset$. Thus $L \cap A \in \mathcal{L} \upharpoonright A$ and $R \cap A \in \mathcal{G}(\mathcal{L}) \upharpoonright A$. In the proof of Proposition 3.1, we showed that $\mathcal{G}(\mathcal{L}) \upharpoonright A \subseteq \mathcal{G}(\mathcal{L} \upharpoonright A)$. Thus $R \cap A \in \mathcal{G}(\mathcal{L} \upharpoonright A)$. The set B is a subalphabet of $\mathcal{L} \upharpoonright A$, so

$$\forall L' \in \mathcal{L} \upharpoonright A \quad \forall R' \in \mathcal{G}(\mathcal{L} \upharpoonright A)\ [L' \cap B \neq \emptyset \ \wedge\ R' \cap B \neq \emptyset \Rightarrow L' \cap R' \subseteq B].$$

Thus $(L \cap A) \cap (R \cap A) \subseteq B$. But $(L \cap A) \cap (R \cap A) = (L \cap R) \cap A = L \cap R$ (because $L \cap R \subseteq A$, which, in turn, is true because $L \cap A \neq \emptyset$, $R \cap A \neq \emptyset$, and A is a subalphabet of $\mathcal{L}$). So $L \cap R \subseteq B$. Q.E.D.

LEMMA 3.3: If A_i is a subalphabet of $\mathcal{L}$ for each i, then $\bigcap_i A_i$ is a subalphabet of $\mathcal{L}$.

PROOF: Let $L \cap \bigcap_i A_i \neq \emptyset$ and $R \cap \bigcap_i A_i \neq \emptyset$. Then, for every i, $L \cap A_i \neq \emptyset$ and $R \cap A_i \neq \emptyset$. Then (since A_i is a subalphabet of $\mathcal{L}$), for every i, $L \cap R \subseteq A_i$. So $L \cap R \subseteq \bigcap_i A_i$. Q.E.D.

DEFINITION: The set A is a <u>direct subalphabet</u> of $\mathcal{L}$ iff both A and Alph($\mathcal{L}$) - A are subalphabets of $\mathcal{L}$.

PROPOSITION 3.4: If A is a direct subalphabet of $\mathcal{L}$, then either

$$\forall L \in \mathcal{L}\ (L \subseteq A \vee L \subseteq \text{Alph}(\mathcal{L}) - A), \text{ or}$$

$$\forall R \in \mathcal{G}(\mathcal{L})\ (R \subseteq A \vee R \subseteq \text{Alph}(\mathcal{L}) - A).$$

PROOF: Assume that there is a set L in $\mathcal{L}$ such that $L \cap A \neq \emptyset$ and $L - A \neq \emptyset$. Assume also that there is a set R in $\mathcal{G}(\mathcal{L})$ such that $R \cap A \neq \emptyset$ and $R - A \neq \emptyset$. Since $L \cap A \neq \emptyset$ and $R \cap A \neq \emptyset$ we have that $L \cap R \subseteq A$. But since $L - A \neq \emptyset$ and $R - A \neq \emptyset$ we have $L \cap R \subseteq \text{Alph}(\mathcal{L}) - A$. This is a contradiction. Q.E.D.

COROLLARY: If A is a direct subalphabet of $\mathcal{L}$, then $\mathcal{L} = \mathcal{L}_A \dot{\vee} \mathcal{L}_{\text{Alph}(\mathcal{L})-A}$ or $\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{L}_A \dot{\wedge} \mathcal{L}_{\text{Alph}(\mathcal{L})-A}$, where $\text{Alph}(\mathcal{L}_A) = A$.

PROOF: Case(1) For every L in $\mathcal{L}$, $L \subseteq A$ or $L \subseteq \text{Alph}(\mathcal{L}) - A$. Then $\mathcal{L} = \mathcal{L}_A \dot{\vee} \mathcal{L}_{\text{Alph}(\mathcal{L})-A}$.

Case(2) For every R in $\mathcal{G}(\mathcal{L})$, $R \subseteq A$ or $R \subseteq \text{Alph}(\mathcal{L}) - A$. Then $\mathcal{R} = \mathcal{R}_A \dot{\vee} \mathcal{R}_{\text{Alph}(\mathcal{L})-A}$. So by Lemma 2.3 (a) $\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{G}(\mathcal{R}) = \mathcal{G}(\mathcal{R}_A) \dot{\wedge} \mathcal{G}(\mathcal{R}_{\text{Alph}(\mathcal{L})-A})$. Let $\mathcal{L}_A$ be $\mathcal{G}(\mathcal{R}_A)$, and let $\mathcal{L}_{\text{Alph}(\mathcal{L})-A}$ be $\mathcal{G}(\mathcal{R}_{\text{Alph}(\mathcal{L})-A})$. Q.E.D.

PROPOSITION 3.5: If A_i is a direct subalphabet of $\mathcal{L}$ for each i, then $\bigcup_i A_i$ is a subalphabet of $\mathcal{L}$.

PROOF: Let A_i be a direct subalphabet of $\mathcal{L}$ for each i.

Case(1) Let $\mathcal{L} = \mathcal{L}_{A_0} \dot{\vee} \mathcal{L}_{\text{Alph}(\mathcal{L})-A_0}$. Claim: For each i, $\mathcal{L} = \mathcal{L}_{A_i} \dot{\vee} \mathcal{L}_{\text{Alph}(\mathcal{L})-A_i}$. Otherwise there is an i such that $\mathcal{G}(\mathcal{L}) = \mathcal{R}_{A_i} \dot{\vee} \mathcal{R}_{\text{Alph}(\mathcal{L})-A_i}$. But by Lemma 2.3 we already have that $\mathcal{G}(\mathcal{L}) = \mathcal{R}_{A_0} \dot{\wedge} \mathcal{R}_{\text{Alph}(\mathcal{L})-A_0}$ and this contradicts Lemma 2.2.

So, for each i, we have $\mathcal{L} = \mathcal{L}_{A_i} \dot{\vee} \mathcal{L}_{\text{Alph}(\mathcal{L})-A_i}$. This means that, for each L in $\mathcal{L}$, either $L \subseteq A_i$ or $L \subseteq \text{Alph}(\mathcal{L}) - A_i$, for each i. So, for each L in $\mathcal{L}$, either $L \subseteq \bigcup A_i$ or $L \subseteq \text{Alph}(\mathcal{L}) - \bigcup A_i$. This makes $\bigcup A_i$ a direct subalphabet of $\mathcal{L}$. This is because

(a) if $L \cap \bigcap A_i \neq \emptyset$ then $L \subseteq \bigcap A_i$, so

$$L \cap \bigcup A_i \neq \emptyset \wedge R \cap \bigcup A_i \neq \emptyset \Rightarrow L \cap R \subseteq \bigcup A_i;$$

and

(b) if $L - \bigcup A_i \neq \emptyset$ then $L \subseteq \mathrm{Alph}(\mathcal{L}) - \bigcup A_i$, so

$$L - \bigcup A_i \neq \emptyset \wedge R - \bigcup A_i \neq \emptyset \Rightarrow L \cap R \subseteq \mathrm{Alph}(\mathcal{L}) - \bigcup A_i.$$

Case(2) Let $\mathcal{G}(\mathcal{L}) = \mathcal{R}_{A_0} \dot{\vee} \mathcal{R}_{\mathrm{Alph}(\mathcal{L})-A_0}$. This case is handled similarly. Q.E.D.

COROLLARY: If A_i is a direct subalphabet of $\mathcal{L}$ for each i, then $\bigcap_i A_i$ is a direct subalphabet of $\mathcal{L}$.

PROOF: Assume that A_i is a direct subalphabet of $\mathcal{L}$ for each i. By Lemma 3.3, $\bigcap A_i$ is a subalphabet. By Proposition 3.5, $\bigcup(\mathrm{Alph}(\mathcal{L})-A_i)$ is a subalphabet. So $\mathrm{Alph}(\mathcal{L}) - (\bigcap A_i)$ is a subalphabet. Thus $\bigcap A_i$ is a direct subalphabet of $\mathcal{L}$. Q.E.D.

DEFINITION: A direct subalphabet, A, of $\mathcal{L}$ is called <u>proper</u> iff $A \neq \emptyset$ and $A \neq \mathrm{Alph}(\mathcal{L})$.

PROPOSITION 3.6: If $\mathcal{L}$ has a proper direct subalphabet, then $\mathcal{L}$ has a minimal (by the inclusion relation) proper direct subalphabet.

PROOF: Let $\mathfrak{A}$ be the family of proper direct subalphabets of $\mathcal{L}$. Order $\mathfrak{A}$ by the relation $A_1 \leq A_2$ iff $A_1 \supseteq A_2$. The family $\mathfrak{A}$ is non-empty. By the Corollary to Proposition 3.5, every chain of sets in $\mathfrak{A}$ has an upper bound in $\mathfrak{A}$. Therefore, by Zorn's Lemma, $\mathfrak{A}$ has a maximal element. Q.E.D.

The combined effect of these propositions is to show a correspondence between certain components of $\mathcal{L}$ and the weaves that appear on the nodes of the tree for $\mathcal{L}$. The advantage of working with subalphabets rather than components is that the only important relationship between subalphabets is set inclusion (rather than conjunction, disjunction, and what would be a very clumsy notion of "limit" for components.)

Now we can state and prove the following.

THEOREM 3.7: Let $<\mathcal{L}, \mathcal{R}>$ be decomposable and stopped (with $\mathcal{R} = \mathcal{G}(\mathcal{L})$). Then $\mathcal{L} = \{\hat{\sigma} | \sigma$ is a I-strategy on the tree for $\mathcal{L}\}$, and $\mathcal{R} = \{\hat{\tau} | \tau$ is a II-strategy on the tree for $\mathcal{L}\}$.

PROOF: (Remark: Many details of the proof are skipped.)

(a) Let $L \in \mathcal{L}$. We must define σ_L by induction, so as to insure that, for each node $\mathcal{L}'$ in the range of σ_L, $\text{Alph}(\mathcal{L}') \cap L \neq \emptyset$.

Stage 0: Assume that $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$. Assume without loss of generality that $L \in \mathcal{L}_1$. Then let $\sigma_L(\mathcal{L}) = \mathcal{L}_1$. Since $L \in \mathcal{L}_1$, $L \cap \text{Alph}(\mathcal{L}_1) \neq \emptyset$.

Stage β: Assume, by the hypothesis of induction, that $L \cap \text{Alph}(\mathcal{L}^{\beta}) \neq \emptyset$. Let $\sigma_L(\mathcal{L}^{\beta})$ be all and only successor nodes, $\mathcal{L}_i^{\beta+1}$, of $\mathcal{L}^{\beta}$ satisfying $L \cap \text{Alph}(\mathcal{L}_i^{\beta+1}) \neq \emptyset$. (It can be checked that if β is of odd parity, all successors of $\mathcal{L}^{\beta}$ will satisfy this condition, and if β is of even parity, then one and only one successor of $\mathcal{L}^{\beta}$ will satisfy the condition.)

We must show that $\hat{\sigma}_L = L$.

$\supseteq$: For any α, let $(\hat{\sigma}_L)_\alpha$ be the union of all alphabets on nodes of level α that appear in σ_L. Since $\sigma_L(\mathcal{L}^\beta)$ is all and only successors, $\mathcal{L}_i^{\beta+1}$, of $\mathcal{L}^\beta$ satisfying $L \cap \mathrm{Alph}(\mathcal{L}_i^{\beta+1}) \neq \emptyset$, we have, for any α, $L \subseteq (\hat{\sigma}_L)_\alpha$. Therefore $L \subseteq \hat{\sigma}_L$ (because $\hat{\sigma}_L = \bigcap_\alpha (\hat{\sigma}_L)_\alpha$).

$\subseteq$: Assume that $d \in \mathrm{Alph}(\mathcal{L})$ and $d \notin L$. Let n be the node of lowest level that is labeled with {d}. (Remark: At this point in the proof we must recognize the distinction between nodes on the tree and their labels.)

Case (1) The node n is a successor node. Let m be the predecessor of n. Even if m is in the domain of σ_L, we cannot have $n \in \sigma_L(m)$, because $\{d\} \cap L = \emptyset$. So n is not in the domain of σ_L, and thus d is not in $\hat{\sigma}_L$. (Remark: For any β, the alphabets on distinct nodes of level β are pairwise disjoint.)

Case (2) The node n is a limit node. Assume the worst possible situation: that, for each predecessor node n_i of n, we have that n_i is in the domain of σ_L. Then d is in $\hat{\sigma}_L$. So we must show that d is in L. This is done by the following important lemma.

LEMMA 3.8 (The key lemma for subalphabets): Let $A_0 \supseteq A_1 \supseteq \ldots, \ldots \supseteq A_\beta \supseteq \ldots$, $(\beta < \lambda)$ be subalphabets of $\mathcal{L}$. Let L be in $\mathcal{L}$ and assume that $L \cap A_\beta \neq \emptyset$ for each $\beta < \lambda$. Let $\bigcap_{\beta<\lambda} A_\beta \neq \emptyset$. Then $L \cap \bigcap_{\beta<\lambda} A_\beta \neq \emptyset$.

This Lemma insures that, in our case, $L \cap \{d\} \neq \emptyset$, so that d is in L.

PROOF: Case (1) There is a set R in $\mathcal{G}(\mathcal{L})$ such that $R \cap \bigcap_{\beta<\lambda} A_\beta \neq \emptyset$. Then, for each $\beta < \lambda$, we have $L \cap A_\beta \neq \emptyset$ and $R \cap A_\beta \neq \emptyset$, so that $L \cap R \subseteq A_\beta$, for each β. Since $\overline{\overline{L \cap R}} = 1$ and the sets A_β are nested, there must be a letter a such that $L \cap R = \{a\}$ and $a \in A_\beta$ for each β. Thus $a \in L \cap \bigcap_{\beta<\lambda} A_\beta$ so $L \cap \bigcap_{\beta<\lambda} A_\beta \neq \emptyset$.

Case (2) For every set R in $\mathcal{G}(\mathcal{L})$, $R \cap \bigcap_{\beta<\lambda} A_\beta = \emptyset$. This is impossible, because it means that no elements of $\bigcap_{\beta<\lambda} A_\beta$ are in any good choice sets of $\mathcal{L}$, contradicting the fact that $\mathcal{L}$ is a weave (condition (2)) and that $\bigcap_{\beta<\lambda} A_\beta$ is non-empty. Q.E.D.

This completes part (a) of the proof. Now we know that $\mathcal{L} \subseteq \{\hat{\sigma} | \sigma$ is a I-strategy on the tree for $\mathcal{L}\}$.

(b) The map $L \to \sigma_L$, from sets in $\mathcal{L}$ to I-strategies, is one-to-one and onto. If $L_1 \neq L_2$, then $\hat{\sigma}_{L_1} \neq \hat{\sigma}_{L_2}$, so $\sigma_{L_1} \neq \sigma_{L_2}$. Thus the map is one-to-one. For every I-strategy σ there is one and only one set L in $\mathcal{L}$ satisfying

$$\forall \alpha \ (\hat{\sigma})_\alpha \cap L \neq \emptyset.$$

The strategy σ is the image of L under the map. Thus $\mathcal{L} = \{\hat{\sigma} | \sigma$ is a I-strategy on the tree for $\mathcal{L}\}$.

(c) and (d) In a similar manner we can do constructions showing that

$\mathcal{R} = \{\hat{\tau} | \tau$ is a II-strategy on the tree for $\mathcal{L}\}$. Q.E.D.

We have proved that every stopped decomposable weave can be represented by a stopped tree. Likewise every stopped tree represents a stopped decomposable weave, and this is stated in the following theorem.

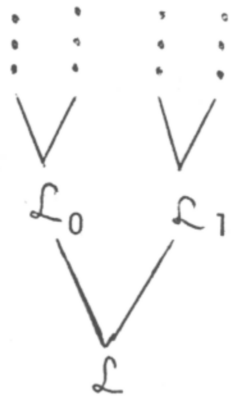
THEOREM 3.9: Let $\mathcal{T}$ be a stopped tree, and let $\mathcal{L} = \{\hat{\sigma} | \sigma$ is a I-strategy on $\mathcal{T}\}$ and $\mathcal{R} = \{\hat{\tau} | \tau$ is a II-strategy on $\mathcal{T}\}$. Then $< \mathcal{L}, \mathcal{R} >$ is a stopped decomposable weave, where $\mathcal{L} = \mathcal{G}(\mathcal{R})$ and $\mathcal{R} = \mathcal{G}(\mathcal{L})$.

PROOF: (sketch) Properties (1) and (2) in the definition of weave are well known properties of strategies for games. The proof that $\mathcal{L} = \mathcal{G}(\mathcal{R})$ and that $\mathcal{R} = \mathcal{G}(\mathcal{L})$ is similar to the construction in the previous theorem.

If the tree begins with

,

then we can label the lower nodes as follows.



Here $\mathcal{L} = \mathcal{L}_0 \dot{\vee} \mathcal{L}_1$, where $\mathcal{L}_0 = \{\hat{\sigma}|\sigma$ is a I-strategy on $\mathcal{T}$ that requires player I to move to the left at stage 0}, and $\mathcal{L}_1 = \{\hat{\sigma}|\sigma$ is a I-strategy on $\mathcal{T}$ that requires player II to move to the right at stage 0}. Like wise, $\mathcal{L}_0$ and $\mathcal{L}_1$ can be decomposed, and so on. Q.E.D.

COROLLARY: Let $< \mathcal{L}, \mathcal{R} >$ be decomposeable and stopped, and assume that $\mathcal{R} = \mathcal{G}(\mathcal{L})$. Then $\mathcal{L} = \mathcal{G}(\mathcal{R})$.

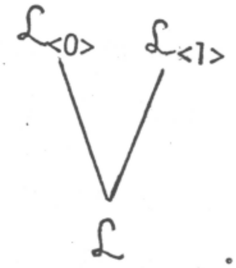
PROOF: Form the tree for $\mathcal{L}$ as in Theorem 3.7. Then apply Theorem 3.9. Q.E.D.

Throughout this section we've been dealing with trees of possibly infinite ordinal length. It might be helpful to give an example of a weave whose decomposition involves more than ω steps.

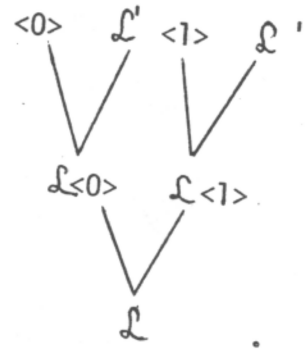
First we construct a weave $\mathcal{L}$ whose alphabet is $2^{\underline{\omega}}$ --the set of finite sequences of zeroes and ones. The sets in $\mathcal{L}$ are indexed by 2^{ω}--the set of sequences (of zeroes and ones) of length ω. Let $a \in 2^{\underline{\omega}}$ and $s \in 2^{\omega}$. The letter a will be a member of the set L_s if and only if a is an initial segment of s. For instance we have the set

$$L_{<0,1,0,1,\ldots>} = \{<0>, <0,1>, <0,1,0>, \ldots\}.$$

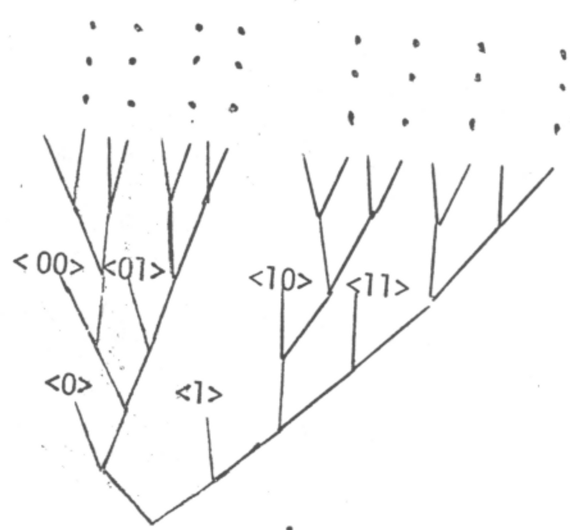
Let $\mathcal{L}_{<0>}$ be the subfamily of $\mathcal{L}$ containing all sets L_s such that s begins with a zero. Similarly define $\mathcal{L}_{<1>}$. Then $\mathcal{L} = \mathcal{L}_{<0>} \dot{\vee} \mathcal{L}_{<1>}$, so we have



Each set in $\mathcal{L}_{<0>}$ contains the letter <0>, so we can factor <0> out. Thus $\mathcal{L}_{<0>} = <0> \dot{\wedge} \mathcal{L}'$. Likewise $\mathcal{L}_{<1>} = <1> \dot{\wedge} \mathcal{L}''$. So we have

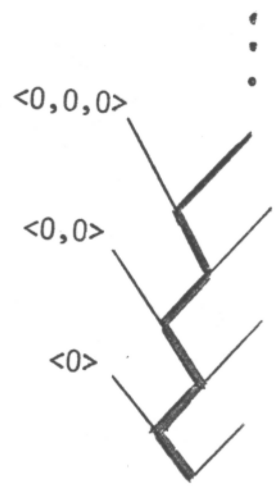


The families $\mathcal{L}'$ and $\mathcal{L}''$ can be decomposed in a similar manner. By continuing this way we get the tree



This tree is continuing, and the level of each node that is not labeled with " Λ " is a finite ordinal. (Notice that $\mathcal{L}$ is normal.)

We can change $\mathcal{L}$ into $\mathcal{L}^*$ by adding a letter a* to the alphabet, and saying that a* is in $L_{<0,0,0,\ldots>}$ (and <u>only</u> in $L_{<0,0,0,\ldots>}$). The tree for $\mathcal{L}^*$ is the same as the tree for $\mathcal{L}$ except that it has a node labeled with {a*} at the ωth level. This node appears at the top of the branch pictured below.



(The weave $\mathcal{L}^*$ is also normal.)

By adding to $\mathcal{L}$ a letter for each element of 2^ω, and putting each letter at the top of the appropriate branch, we can turn $\mathcal{L}$ into a stopped weave.

In order to get nodes of levels higher than ω, we can add letters to $\mathcal{L}$ in a more complicated form. For instance, in our original definition of $\mathcal{L}$, replace the set $L_{<0,0,0,\ldots>}$ by the sets

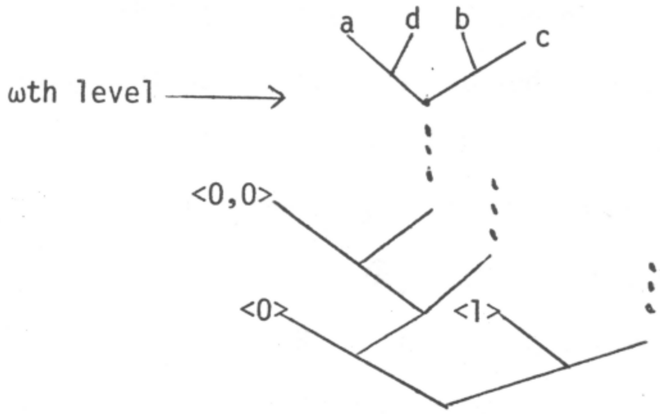
$$L_1 = L_{<0,0,0,\ldots>} \cup \{a,b\},$$

$$L_2 = L_{<0,0,0,\ldots>} \cup \{c,d\},$$

$$L_3 = L_{<0,0,0,\ldots>} \cup \{a,c\}, \text{ and}$$

$$L_4 = L_{<0,0,0,\ldots>} \cup \{b,d\}.$$

Then the tree for this new weave is



Theorems 3.7 and 3.9 dealt with stopped weaves. The situation for continuing weaves is a bit more complicated. Let $<\mathcal{L}, \mathcal{R}>$ be a continuing decomposable weave, where $\mathcal{R} = \mathcal{G}(\mathcal{L})$. Form $\mathcal{T}$, the tree for $\mathcal{L}$. As in the proof for stopped weaves, every set L in $\mathcal{L}$ is equal to some $\hat{\sigma}$, where σ is a I-strategy in $\mathcal{T}$. But now not every $\hat{\sigma}$ is a set in $\mathcal{L}$.

For instance, let b be a continuing branch, and assume that there is a II-strategy τ such that $\hat{\tau}$ is in $\mathcal{R}$, and τ passes through b. Then there can be no I-strategy σ such that $\hat{\sigma}$ is in $\mathcal{L}$, and σ passes through b, because if so, then the resulting play (player I playing σ and player II playing τ) would be the branch b, which has no non-empty label on its final node, so $\hat{\sigma} \cap \hat{\tau} = \emptyset$.

A careful analysis will reveal some of the pattern governing strategies and sets in $\mathcal{L}$.

Let $<\mathcal{L}, \mathcal{R}>$ be a decomposable weave with $\mathcal{R} = \mathcal{G}(\mathcal{L})$. Form $\mathcal{T}$, the tree for $\mathcal{L}$. Let b be a continuing branch on $\mathcal{T}$ and label b with

(1) "$\neg \exists$ L, $\exists$ R", if there is no I-strategy σ such that $\hat{\sigma} \in \mathcal{L}$ and σ passes through b, and there exists a II-strategy τ such that $\hat{\tau} \in \mathcal{R}$ and τ passes through b;

(2) "$\exists$ L, $\neg \exists$ R", if there exists a I-strategy σ such that $\hat{\sigma} \in \mathcal{L}$ and σ passes through b, and there is no II-strategy τ such that $\hat{\tau} \in \mathcal{R}$ and τ passes

through b; or

(3) "$\neg \exists$ L, $\neg \exists$ R" if there is no I-strategy σ such that $\hat{\sigma} \in \mathcal{L}$ and σ passes through b, and there is no II-strategy τ such that $\hat{\tau} \in \mathcal{R}$ and τ passes through b.

Notice that the discussion of two paragraphs above precludes the possibility of having continuing branches labeled "$\exists$ L, $\exists$ R". We will show that no branches can be labeled "$\neg \exists$ L, $\neg \exists$ R" either.

Let $\mathcal{L}' = \{\hat{\sigma} |$ the only continuing branches that σ passes through are labeled "$\exists$ L, $\neg \exists$ R"$\}$. Let $\mathcal{R}' = \{\hat{\tau} | \tau$ passes through no continuing branches labeled "$\exists$ L, $\neg \exists$ R". If $L \in \mathcal{L}$, then $L = \hat{\sigma}$ for some I-strategy σ and certainly cannot pass through a continuing branch labeled "$\neg \exists$ L, $\exists$ R", so $L \in \mathcal{L}'$. Therefore $\mathcal{L} \subseteq \mathcal{L}'$.

Now we must show that $\mathcal{R}' \subseteq \mathcal{G}(\mathcal{L}') \subseteq \mathcal{G}(\mathcal{L})$. The second inclusion follows from the fact that $\mathcal{L} \subseteq \mathcal{L}'$. Let $R' \in \mathcal{R}'$. Then $R' = \hat{\tau}$, for some τ that doesn't pass through a continuing branch labeled "$\exists$ L, $\neg \exists$ R". If $L \in \mathcal{L}$, then $L = \hat{\sigma}$, for some σ that passes through no continuing branches labeled "$\neg \exists$ L, $\exists$ R" or "$\neg \exists$ L, $\neg \exists$ R". So the play resulting from σ and τ must be a stopped branch. So $\hat{\sigma} \cap \hat{\tau}$ is non-empty. So $\hat{\sigma} \cap \hat{\tau}$ is a singleton. So $R' \cap L$ is a singleton. Similarly, if $L' \in \mathcal{L}'$, then $R' \cap L'$ is a singleton.

Now we want to show that $\mathcal{G}(\mathcal{L}) \subseteq \mathcal{R}'$. Assume not. Then there is a set G in $\mathcal{G}(\mathcal{L})$ such that $G \notin \mathcal{R}'$; i.e, G is equal to $\hat{\tau}$, and τ passes through some branch labeled "$\exists$ L, $\neg \exists$ R". Choose a set L in $\mathcal{L}$ such that $L = \hat{\sigma}$ and σ passes through that branch. Then the play resulting from σ and τ is that branch, so $\hat{\sigma} \cap \hat{\tau} = \emptyset$. Thus $L \cap G = \emptyset$, so G is not a gcs for $\mathcal{L}$. This is a contradiction. Therefore $\mathcal{R}' = \mathcal{G}(\mathcal{L}') = \mathcal{G}(\mathcal{L})$.

To prove that $\mathcal{L} = \mathcal{L}'$, we need a construction similar to that in the proof of Theorem 3.7 and a demonstration that the map $L \to \sigma_L$ is onto the set of I-strategies σ that satisfy the condition given in the definition of $\mathcal{L}'$. The details can be recovered from the proof of Theorem 3.7 and so will be omitted.

To summarize what we have done thus far, we begin with $<\mathcal{L},\mathcal{R}>$ and use it to form the tree $\mathcal{T}$. We label some branches with "$\exists$L, $\neg\exists$R", meaning that they are off limits to II-strategies τ, whose $\hat{\tau}$ is in $\mathcal{R}$, and that they <u>are</u> to be used by some I-strategies σ, whose $\hat{\sigma}$ is in $\mathcal{L}$.

It turns out that $\mathcal{L}$ is equal to the set of all $\hat{\sigma}$ such that σ passes through only these continuing branches labeled "$\exists$ L, $\neg\exists$ R", and that $\mathcal{R}$ is equal to the set of all $\hat{\tau}$ such that τ passes through none of these branches.

Notice that we didn't have to say that, in order for $\hat{\tau}$ to be in $\mathcal{R}$, τ must not pass through a branch labeled "$\neg\exists$ L, $\neg\exists$ R". This means that if τ doesn't pass through any branches labeled "$\exists$ L, $\neg\exists$ R", then it doesn't pass thru any branches labeled "$\neg\exists$ L, $\neg\exists$ R".

So let b be a branch labeled "$\neg\exists$ L, $\neg\exists$ R". We will show that there is a node n on b such that every I-strategy σ, starting from n, passes through a branch labeled "$\neg\exists$ L, $\exists$ R" or "$\neg\exists$ L, $\neg\exists$ R". Assume otherwise: that for every node n on b there is a I-strategy σ starting from n that passes thru no branch labeled "$\neg\exists$ L, $\exists$ R" or "$\neg\exists$ L, $\neg\exists$ R". Using this information we will construct a strategy σ^* and make sure that it passes through no node labeled "$\neg\exists$ L, $\exists$ R" or "$\neg\exists$ L, $\neg\exists$ R" <u>other than b</u>, and it <u>will</u> pass through b.

To construct σ^*, let x_0 be the initial node of the tree, and define σ^* on x_0 so that $\sigma^*(x_0)$ is a node on b. (Let $\sigma^*(x_0) = k$.) Let n_0 and n_1 be the direct successors of k, and assume that n_0 is not on b and n_1 is on b. Since there is a I-strategy σ starting from k that passes through no node labeled " $\neg \exists$ L, $\exists$ R" or " $\neg \exists$ L, $\neg \exists$ R", we take the part of σ that passes through n_0 and attach it onto σ^* (i.e., the strategies σ^* and σ are the same from n_0 on.) As for n_1, define $\sigma(n_1)$ so that $\sigma(n_1)$ is a node on b. Continue by induction.

Now this strategy σ^* violates our promise that every I-strategy passing through a " $\neg \exists$ L, $\neg \exists$ R"--branch must pass through at least one " $\neg \exists$ L, $\exists$ R"--branch. So we have a contradiction. Therefore, there is a node n on b such that every strategy σ starting from n passes through a branch labeled " $\neg \exists$ L, $\exists$ R" or " $\neg \exists$ L, $\neg \exists$ R".

Let A be the subalphabet of Alph($\mathcal{L}$) that is used to label this node n. For every set L in $\mathcal{L}$, L $\cap$ A = $\emptyset$, because, for every σ satisfying $\hat{\sigma} \varepsilon \mathcal{L}$, σ does not pass through n. So the letters that appear in A are not in Alph($\mathcal{L}$). This is impossible.

Therefore, there is no branch b on the tree for $\mathcal{L}$ such that b is labeled " $\neg \exists$ L, $\neg \exists$ R".

From the above discussion we extract the essential parts of the proof of the following theorem, which is a tree theorem for continuing weaves.

THEOREM 3.10: Let $\mathcal{T}$ be a tree with every continuing branch labeled either " $\exists$ L, $\neg \exists$ R" or " $\neg \exists$ L, $\exists$ R". Let $\mathcal{L} = \{\hat{\sigma} | \sigma$ is a I-strategy that passes through no branches labeled " $\neg \exists$ L, $\exists$ R"$\}$, and let $\mathcal{R} = \{\hat{\tau} | \tau$ is a II-strategy that passes through no branches labeled " $\exists$ L, $\neg \exists$ R" . Then $< \mathcal{L}, \mathcal{R} >$ is a decomposable weave, where $\mathcal{L} = \mathcal{G}(\mathcal{R})$ and $\mathcal{R} = \mathcal{G}(\mathcal{L})$. Conversely, every continuing decomposable weave comes from such a tree.

From now on, when we say "let $\mathcal{T}$ be a continuing tree", we will assume that every continuing branch on $\mathcal{T}$ is labeled with either "$\neg \exists$ L, $\exists$ R" or "$\exists$ L, $\neg \exists$ R".

We end this section with a definition.

DEFINITION: For a continuing tree $\mathcal{T}$, a I-strategy is called admissible iff it passes through no branches labeled "$\neg \exists$ L, $\exists$ R". A II-strategy is admissible iff it passes through no branches labeled "$\exists$ L, $\neg \exists$ R".

SECTION 4: STATEMENTS

In this section we show how certain weaves can be represented by (certain) statements in logic. We will use the propositional calculus, the infinitary homogeneous predicate calculus, and the infinitary heterogeneous predicate calculus.

Let ψ be a statement in the propositional calculus, using only the connectives $\wedge$ and $\vee$, and let no atomic formula occur more than once in ψ. It can be shown that ψ represents a normal weave. For instance, let ψ be $(a \vee b) \wedge (c \vee d)$. Using the associative law, we can write ψ in the form $(a \wedge c) \vee (a \wedge d) \vee (b \wedge c) \vee (b \wedge d)$. This represents the family $\{\{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}\}$. Let this be $\mathcal{L}$. By reversing the connectives of ψ we get $(a \wedge b) \vee (c \wedge d)$, which represents $\{\{a,b\}, \{c,d\}\}$. Let this family be $\mathcal{R}$. Notice that $\mathcal{R} = \mathcal{G}(\mathcal{L})$, and $\mathcal{L} = \mathcal{G}(\mathcal{R})$.

THEOREM 4.1: Let ψ be a statement in the propositional calculus, using only the connectives $\wedge$ and $\vee$, and let no atomic formula occur more than once in ψ. Then ψ represents a normal weave $\mathcal{L}$, where Alph($\mathcal{L}$) is finite, and the dual of ψ represents $\mathcal{G}(\mathcal{L})$.

The proof is by induction on the number of connectives in ψ.

The converse of Theorem 4.1 is given by the Fundamental Theorem of Normal Weaves. Let $\mathcal{L}$ be a normal weave with $\overline{\overline{\text{Alph}(\mathcal{L})}} = n$, where n is a finite number. If $n = 1$, then $\mathcal{L} = \{\{a\}\}$, so $\mathcal{L}$ is represented by the statement a.

If $n > 1$, then, by the Fundamental Theorem, we have that $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$ or $\mathcal{L} = \mathcal{L}_1 \dot{\wedge} \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both normal. By the hypothesis of induction, we have statements ψ_1 and ψ_2 to represent $\mathcal{L}_1$ and $\mathcal{L}_2$, and so $\psi_1 \vee \psi_2$ (or $\psi_1 \wedge \psi_2$) represents $\mathcal{L}$.

THEOREM 4.2: Every normal weave $\mathcal{L}$, where Alph($\mathcal{L}$) is finite, can be represented by a statement in the propositional calculus.

What if Alph($\mathcal{L}$) is not finite? Let ψ be the statement $\exists x_0 x_1 \ldots \forall y P(y, x_0, x_1, \ldots)$ in the infinitary homogeneous predicate calculus. We can rewrite ψ, roughly, as $\bigvee_{x_0} \bigvee_{x_1} \ldots \bigwedge_{y} P(y, x_0, x_1, \ldots)$. This represents the weave $\mathcal{L} = \{\{P(b^0, a_0, a_1, \ldots), P(b^1, a_0, a_1, \ldots), \ldots\}, \{P(b^0, a_0', a_1', \ldots), P(b^1, a_0', a_1', \ldots), \ldots\}, \ldots\}$.

Here $b^0, b^1, \ldots$ are all possible values of y, and $a_0^{''\cdots'}, a_1^{''\cdots'}, \ldots$ is taken to be a sequence of values for $x_0, x_1, \ldots$.

THEOREM 4.3: Let ψ be a statement in the infinitary homogeneous predicate calculus, using only $\wedge$, $\vee$, $\forall$, and $\exists$, and let no atomic formula occur more than once in ψ. Then ψ represents a normal weave $\mathcal{L}$, and the dual of ψ represents $\mathcal{G}(\mathcal{L})$. Every branch on the tree for $\mathcal{L}$ is finite.

PROOF: Define the complexity of a statement as follows.

(1) If ψ is atomic, then $c(\psi) = 0$.

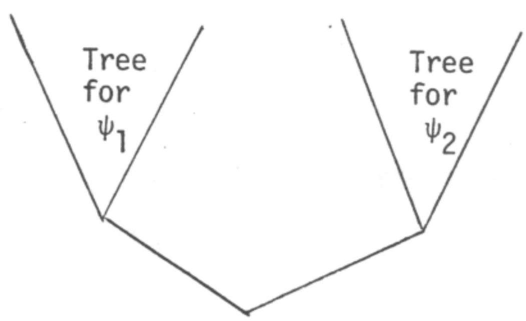
(2) If ψ is $\psi_1 \vee \psi_2$ or ψ is $\psi_1 \wedge \psi_2$, then $c(\psi) = c(\psi_1) + c(\psi_2) + 1$.

(3) If ψ is $\forall x_0 \forall x_1, \ldots \Psi(x_0,x_1,\ldots)$ or ψ is $\exists x_0 \exists x_1 \ldots$ $\Psi(x_0,x_1,\ldots)$, then $c(\psi) = \sum_{x_0,x_1,\ldots} c(\Psi(x_0,x_1,\ldots)) + 1$.

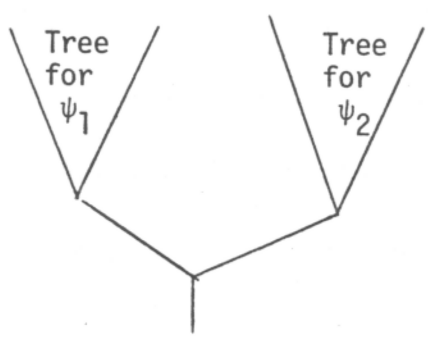
The theorem is proved by induction on the complexity of ψ. Here we prove only that every branch on the tree is finite.

If ψ is atomic, then the tree for $\mathcal{L}$ consists of one point, so we are done.

If ψ is $\psi_1 \vee \psi_2$, then the tree for ψ is of the following form.

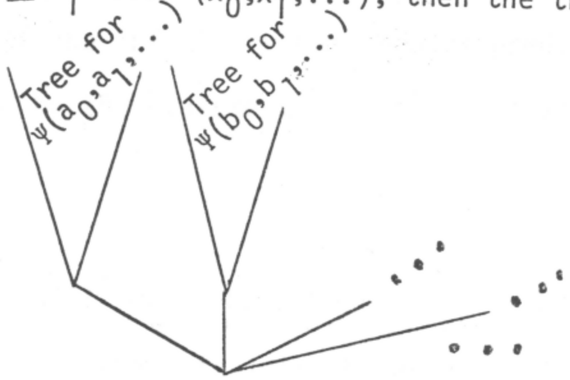


By the hypothesis of induction, all branches on this tree are finite. If ψ is $\psi_1 \wedge \psi_2$, then the tree for ψ is of the following form.



All branches on this tree are finite.

If ψ is $\exists x_0 \exists x_1 \ldots \Psi(x_0, x_1, \ldots)$, then the tree for ψ is



By the hypothesis of induction, all branches on this tree are finite. Similarly, for $\forall x_0 \forall x_1 \ldots \Psi(x_0, x_1, \ldots)$, all branches are finite. Q.E.D.

THEOREM 4.4: Let $\mathcal{T}$ be a tree, and let all branches of $\mathcal{T}$ be finite. Then $\mathcal{T}$ is represented by a statement in the infinitary homogeneous predicate calculus.

PROOF: To each node on the tree we assign a statement.

To each final node assign a distinct atomic statement.

Let $\psi_0, \psi_1, \ldots$ be assigned to the successors $n_0, n_1, \ldots$ of n. If n is a node of even parity, assign the statement $\bigvee \psi_i$ to n. If n is a node of odd parity, assign $\bigwedge \psi_i$ to n.

By continuing in this way, we assign a statement to every node. For if the initial node, x_0, is not assigned a statement, then there is some successor node, x_1, of x_0 that is not assigned a statement. Likewise there is a successor, x_2, of x_1 that is not assigned a statement, and so on. By continuing in this way we create a branch $x_0, x_1, \ldots, x_n$ whose topmost node is not assigned a statement. But this is impossible.

We can now rename the atomic statements so that the statements thus assigned are part of the infinitary homogeneous predicate calculus. The tree is represented by the statement that has been assigned to the initial node. Q.E.D.

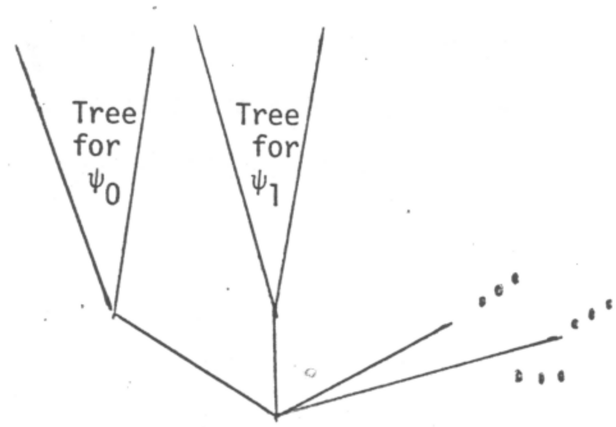
If every branch of $\mathcal{T}$ is finite, then $\mathcal{T}$ is stopped. This is also the case with statements in the infinitary heterogeneous predicate calculus.

THEOREM 4.5: Let ψ be a statement in the infinitary heterogeneous predicate calculus, using only $\wedge$, $\vee$, $\forall$, and $\exists$, and let no atomic formula occur more than once in ψ. Then ψ represents a decomposable weave $\mathcal{L}$, and the dual of ψ represents $\mathcal{G}(\mathcal{L})$. The tree for $\mathcal{L}$ is stopped.

PROOF: We define the complexity of a statement ψ as before, and show by induction that the tree for ψ is stopped.

If ψ is atomic, then the tree for ψ consists of one point.

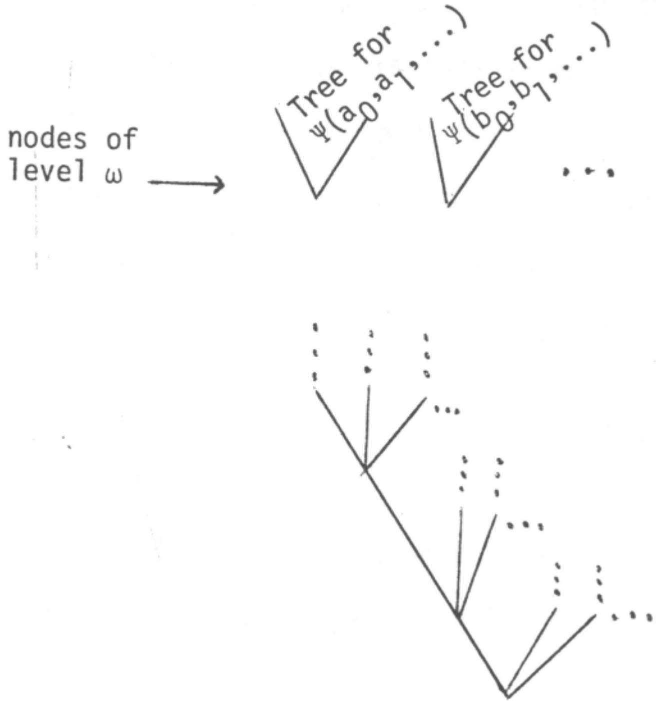
Let ψ be of the form $\bigvee \psi_i$ (or $\exists x_0 \exists x_1 \ldots \Psi(x_0,x_1,\ldots)$). The tree for ψ is of the following form



By the hypothesis of induction, the trees for ψ_0, ψ_1, ... are all stopped, so this tree is stopped.

Let ψ be of the form $\exists x_0 \; \forall x_1 \; \exists x_2 \; \ldots \; \Psi(x_0, x_1, x_2 \ldots)$.

Then the tree for ψ is of the following form.



For each sequence a_0, a_1, ..., the tree for $\Psi(a_0, a_1, \ldots)$ is stopped (by the hypothesis of induction) so the tree for ψ is stopped. Q.E.D.

Let $\mathcal{L}$ be the weave given in the 3rd paragraph on page 43. The tree for this weave is continuing, so this statement cannot be represented by a statement in any of the languages used in this section.

SECTION 5: THE p,q-THEOREM

In this brief section we present an application of the theorems which state that certain weaves can be represented as trees. The p, q-Theorem was first proved by Gaisi Takeuti using different methods.

Let $< \mathcal{L}, \mathcal{R} >$ be a decomposable weave, where $\mathcal{R} = \mathcal{G}(\mathcal{L})$, and let $\mathcal{T}$ be the tree for $\mathcal{L}$. Let p and q be letters in the alphabet of $\mathcal{L}$. Corresponding to p and q there are two (distinct) branches b_p and b_q on $\mathcal{T}$ such that the final node of b_p is labeled {{p}}, and the final node of b_q is labeled {{q}}.

DEFINITION: Let n be a node on $\mathcal{T}$. We say that p and q <u>part company</u> at n iff n is the node of highest level that is on both b_p and b_q.

LEMMA 5.1: The letters p and q part company at a node of odd parity iff there is a set L in $\mathcal{L}$ such that p and q are in L.

PROOF: The proof is given for continuing weaves. A simplification of this proof can be used for stopped weaves.

$\Rightarrow$: Assume that p and q part company at n, a node of odd parity. Since p and q are in Alph($\mathcal{L}$), there are admissible I-strategies σ_p and σ_q such that $b_p \in \hat{\sigma}_p$ and $b_q \in \hat{\sigma}_q$. Let n' be the successor of n that is on the branch b_q. Let σ be the same as σ_p except at n' (and above). At the node n', attach that part of σ_q that begins at n'.

Since σ_p and σ_q are admissible I-strategies, σ_p and σ_q pass through no branches labeled "$\neg\exists$ L, $\exists$ R". Thus σ passes through no branches labeled "$\neg\exists$ L, $\exists$ R". So σ is an admissible I-strategy. Notice also that b_p, $b_q \in \hat{\sigma}$. Therefore, σ represents a set L in $\mathcal{L}$ such that p, q $\in$ L.

$\Leftarrow$: If p and q part company at a node of even parity, then no I-strategy σ passes through both branches b_p and b_q. Q.E.D.

LEMMA 5.2: The letters p and q part company of a node of even parity iff there is a set R in $\mathcal{R}$ such that p and q are in R.

The proof is similar to the proof of Lemma 5.1.

THEOREM 5.3 (The p,q-Theorem): Let $\mathcal{L}$ be a decomposable weave, and let p and q be in Alph($\mathcal{L}$). Then either there is a set L in $\mathcal{L}$ such that p, q $\in$ L, or there is a set R in $\mathcal{R}$ such that p, q $\in$ R.

PROOF: The letters p and q must part company either at a node of even parity or at a node of odd parity. Q.E.D.

SECTION 6: TENSOR PRODUCTS

In this section we reprove a special case of Theorem 1.10 using tree methods. In fact, the nature of tensor product becomes transparent when we examine its meaning in terms of trees.

We begin with a definition.

DEFINITION: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be trees. If any branch of $\mathcal{T}_1$ ends on a node of odd parity, add one more node at the top of that branch (so that every branch of $\mathcal{T}_1$ ends on a node of even parity). The tree $\mathcal{T}_1 \otimes \mathcal{T}_2$ is formed by adjoining a copy of $\mathcal{T}_2$ at the top of each stopped branch of $\mathcal{T}_1$.

Translating this into the language of logical statements, we get the following type of definition.

DEFINITION: Let ψ_1 and ψ_2 be statements in the propositional calculus, using only $\wedge$ and $\vee$, such that each atomic formula occurs at most once in ψ_1 and at most once in ψ_2. The atomic formulas of $\psi_1 \otimes \psi_2$ will be ordered pairs of atomic formulas of ψ_1 and ψ_2.

Define $\psi_1 \otimes \psi_2$ by double induction.

If ψ_1 and ψ_2 are atomic, let $\psi_1 \otimes \psi_2$ be the pair $\langle\psi_1, \psi_2\rangle$.

If ψ_1 is atomic, let $\psi_1 \otimes (\phi_1 \vee \phi_2)$ be $(\psi_1 \otimes \phi_1) \vee (\psi_1 \otimes \phi_2)$, and let $\psi_1 \otimes (\phi_1 \wedge \phi_2)$ be $(\psi_1 \otimes \phi_1) \wedge (\psi_1 \otimes \phi_2)$.

If ψ_1 is $\theta_1 \vee \theta_2$, let $\psi_1 \otimes \psi_2$ be $(\theta_1 \otimes \psi_2) \vee (\theta_2 \otimes \psi_2)$. If ψ_1 is $\theta_1 \wedge \theta_2$, let $\psi_1 \otimes \psi_2$ be $(\theta_1 \otimes \psi_2) \wedge (\theta_2 \otimes \psi_2)$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be decomposable weaves, and let $\mathcal{T}_1$ be the tree for $\mathcal{L}_1$ and $\mathcal{T}_2$ be the tree for $\mathcal{L}_2$. For simplicity of exposition we assume for the moment that $\mathcal{L}_1$ and $\mathcal{L}_2$ are both stopped. Let $L \in \mathcal{L}_1 \otimes \mathcal{L}_2$. Then L is of the form $L_1{}^f$, where $L_1 \in \mathcal{L}_1$, $f{:}L_1 \to \mathcal{L}_2$, and $L_1{}^f = \{\langle a,b\rangle \mid a \in L_1 \wedge b \in f(a)\}$.

Each branch on the tree $\mathcal{T}_1 \otimes \mathcal{T}_2$ is actually a pair of branches--one branch from $\mathcal{T}_1$ and the other from $\mathcal{T}_2$. A I-strategy on the tree $\mathcal{T}_1 \otimes \mathcal{T}_2$ is obtained by choosing

(1) a I-strategy σ_1 in $\mathcal{T}_1$, and

(2) a I-strategy σ_a of $\mathcal{T}_2$, for each branch a in $\hat{\sigma}_1$.

So $\hat{\sigma}_1 \in \mathcal{L}_1$, and we have a function from $\hat{\sigma}_1$ into the set of I-strategies of $\mathcal{T}_2$.

The set $\hat{\sigma}_1{}^f$ is a I-strategy of $\mathcal{T}_1 \otimes \mathcal{T}_2$. It corresponds to the set L^f of $\mathcal{L}_1 \otimes \mathcal{L}_2$. This gives us the following proposition.

PROPOSITION 6.1: If $\mathcal{T}_1$ is the tree for $\mathcal{L}_1$, and $\mathcal{T}_2$ is the tree for $\mathcal{L}_2$, then $\mathcal{T}_1 \otimes \mathcal{T}_2$ is the tree for $\mathcal{L}_1 \otimes \mathcal{L}_2$.

For the case where either $\mathcal{L}_1$ or $\mathcal{L}_2$ is continuing (or both are continuing) we replace, in the above explanation, the word "branch" by the words "stopped branch", and replace "I-strategy" by "admissible I-strategy".

We have an immediate corollary.

COROLLARY: If $\mathcal{L}_1$ and $\mathcal{L}_2$ are decomposable, then so is $\mathcal{L}_1 \otimes \mathcal{L}_2$.

Now if $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \ldots$ are represented by the trees $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots$ respectively, each of which has only finite branches, then the tensor product of these trees is of length ω and thus is Borel normal (by the result in [4]). In this case "Borel" refers to subsets of branches on the tree $\bigotimes_i \mathcal{T}_i$. In order to say that $\bigotimes_i \mathcal{L}_i$ is Borel normal we must account for subsets of $\prod_i \mathrm{Alph}(\mathcal{L}_i)$. Fortunately, these two sets ($\widehat{\bigotimes_i \mathcal{T}_i}$ and $\prod_i \mathrm{Alph}(\mathcal{L}_i)$) are isomorphic, and their open sets correspond to one another under the isomorphism. So we can conclude the following.

THEOREM 6.2 (special case of Theorem 1.10): For each $i < \omega$, let $< \mathcal{L}_i, \mathcal{R}_i >$ be a normal weave of D_i, and let the tree for $\mathcal{L}_i$ have only finite branches. Let $\tilde{\mathcal{L}} = \mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \ldots$, let $\tilde{\mathcal{R}} = \mathcal{R}_0 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \ldots$, and let $\tilde{D} = D_0 \times D_1 \times D_2 \times \ldots$. Let X be a Borel subset of $\tilde{D}$. Then there exists a set $\tilde{L}$ in $\tilde{\mathcal{L}}$ such that $\tilde{L} \subseteq X$, or there exists a set $\tilde{R}$ in $\tilde{\mathcal{R}}$ such that $\tilde{R} \subseteq \tilde{D} - X$.

If we do not assume that the trees $\mathcal{T}_0, \mathcal{T}_1, \ldots$ have only finite branches, then we can use Theorem 1.10 to get a result about trees. The theorem states that whenever $\mathcal{L}_0, \mathcal{L}_1, \ldots$ are all normal, then $\bigotimes_i \mathcal{L}_i$ is Borel normal, where "Borel" refers to subsets of the set $\prod_i \text{Alph}(\mathcal{L}_i)$. Each letter in $\text{Alph}(\mathcal{L}_i)$ is actually a branch on the tree $\mathcal{T}_i$. So $\prod_i \text{Alph}(\mathcal{L}_i)$ is the same as the set $\prod_i \hat{\mathcal{T}}_i$. We cannot say for sure whether this set is topologically equivalent to the set of branches on $\bigotimes_i \mathcal{T}_i$. So instead of a theorem for Borel sets of branches we get the following.

THEOREM 6.3: If $\mathcal{T}_0, \mathcal{T}_1, \ldots$ are normal, then the tree $\mathcal{T}_0 \otimes \mathcal{T}_1 \otimes \ldots$ is $\mathcal{F}$-normal, where $\mathcal{F}$ is the family of all Borel subsets of $\prod_i \hat{\mathcal{T}}_i$.

SECTION 7: CROSS SETS AND GENERALIZED TREES

In the proof of The Fundamental Theorem of Normal Weaves, the full strength of the normality assumption was not used until the proof of Lemma 2.10. Before Lemma 2.10 we used arguments of the following sort:

"Choose a particular set C. This set C is a choice set for $\mathcal{L}$ but not a good choice set because

(1) a and b are both in C,

(2) a and b 'clash' (that is, there is a set L in $\mathcal{L}$ such that a and b are both in L, so C is not a gcs), and

(3) if either a or b are eliminated from C, then the resulting set is no longer a choice set."

In the proof of Lemma 2.10, the set C that we chose was one where we had a collection of letters $m_0, m_1, m_2, \ldots$, where, for each i, m_i clashed with $m_0, m_1, \ldots, m_{i-1}, m_{i+1}, m_{i+2}, \ldots$. To eliminate the clashes we had to remove not one letter in a pair, but a whole sequence. In this case, unlike the one in the paragraph in quotation marks, removing one letter at a time was safe (i.e., we still had a choice set) but removing the whole infinite sequence of letters was not safe.

We define a condition that is weaker than normality.

DEFINITION: The weave $\mathcal{L}$ is pre-normal iff, for every cs C, if a and b are in C, and there is a set L in $\mathcal{L}$ such that a and b are in L, then either C-{a} is a cs or C-{b} is a cs.

Any normal weave is pre-normal. The converse is not true. For example, let $\mathcal{L}$ be the collection

$$\{s_0, p_0, p_1, p_2, p_3, \ldots\}$$

$$\{s_1, p_1, p_2, p_3, \ldots\}$$

$$\{s_2, p_2, p_3, \ldots\}$$

$$\vdots$$

and let $\mathcal{R}$ be

$$\{p_0, s_1, s_2, s_3, s_4, \ldots\}$$

$$\{p_1, s_2, s_3, s_4, \ldots\}$$

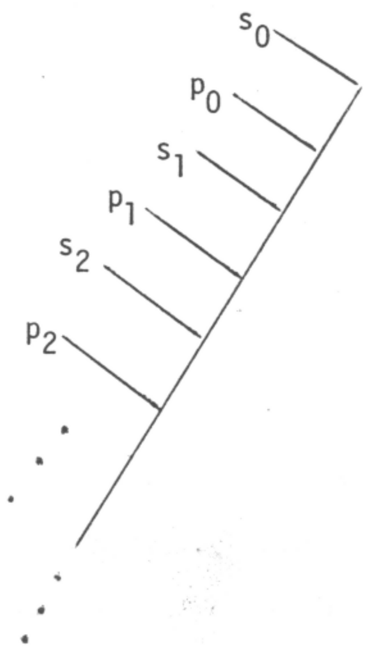
$$\{p_2, s_3, s_4, \ldots\}$$

$$\vdots$$

Then $< \mathcal{L}, \mathcal{R} >$ is a pre-normal weave, but it is not normal. Notice how this example looks when represented as a statement and as a tree. As a statement it has the form

$$[\ldots (((((s_0 \wedge p_0) \vee s_1) \wedge p_1) \vee s_2) \wedge p_2) \vee \ldots],$$

which has no outermost connective. (That is exactly what we wanted to avoid with Lemma 2.10.) As a tree it seems to be represented by the following.



This tree is not well-founded. It seems as if The Fundamental Theorem is proving two things about normal weaves--that they are decomposable, and that the decomposition is well-founded. We can look at this in the language of games. It suggests that what we call "determinateness" is not one property, but actually two properties--decomposability and well-foundedness. Certainly well-foundedness comes into play in determinateness because, in a Gale-Stewart game, determinateness means that, in any choice set of I-strategies, the clashes between plays in this choice set can be eliminated in a well-founded sequence.

Unlike the property of normality, the property of pre-normality can be viewed as a very "local" property of the weave, as is shown in the following set of definitions and lemma.

DEFINITION: Let $\mathcal{L}$ be a weave, and assume that there is a set L in $\mathcal{L}$ such that a, b ε L. Let L_a, L_b ε $\mathcal{L}$, and let a ε L_a, b ε L_b, a $\notin$ L_b, and b $\notin$ L_a. Then $<a,b,L_a,L_b>$ is called a crossed quadruple of $\mathcal{L}$.

DEFINITION: Let $<a,b,L_a,L_b>$ be a crossed quadruple of $\mathcal{L}$. The set X is a cross set for $<a,b,L_a,L_b>$ iff

(1) $X \subseteq L_a \cup L_b - \{a,b\}$,

(2) $X \cap L_a \neq \emptyset$, and

(3) $X \cap L_b \neq \emptyset$.

An example should illustrate the idea more clearly. Let $\mathcal{L}$ be {{x,y,a}, {a,d,b}, {b,u,v}}. Then <a,b,{x,y,a}, {b,u,v}> is a crossed quadruple of $\mathcal{L}$ and {x,y,u,v}, {x,u,v}, {x,y,u}, {x,y,v}, {y,u,v}, {x,u}, {x,v}, {y,u}, and {y,v} are all cross sets for <a,b, {x,y,a}, {b,u,v}>.

DEFINITION: The weave $\mathcal{L}$ is said to have cross sets iff, for every crossed quadruple $<a,b,L_a,L_b>$ of $\mathcal{L}$, there is a set X in $\mathcal{L}$ such that X is a cross set of $<a,b,L_a,L_b>$.

Notice that the weave presented in the example above does not have cross sets. Notice also that {a,b} is a choice set for $\mathcal{L}$, but that neither {a} nor {b} are choice sets for $\mathcal{L}$. Thus $\mathcal{L}$ is not normal.

PROPOSITION 7.1: A weave $\mathcal{L}$ is pre-normal iff it has cross sets.

PROOF: $\Rightarrow$: Assume that $\mathcal{L}$ does not have cross sets. This means that there is a crossed quadruple $\langle a,b,L_a,L_b\rangle$ such that no cross set of $\langle a,b,L_a,L_b\rangle$ is in $\mathcal{L}$. Let $C = [\mathrm{Alph}(\mathcal{L}) - (L_a \cup L_b)] \cup \{a,b\}$. We claim that C is a cs for $\mathcal{L}$, but that neither $C - \{a\}$ nor $C - \{b\}$ is a cs for $\mathcal{L}$.

(1) We show that C is a cs for $\mathcal{L}$. Let $L \in \mathcal{L}$. The set L is not a cross set for $\langle a,b,L_a,L_b\rangle$, so either

(i) $X \not\subseteq L_a \cup L_b - \{a,b\}$,

(ii) $X \cap L_a = \emptyset$, or

(iii) $X \cap L_b = \emptyset$.

If $X \not\subseteq L_a \cup L_b - \{a,b\}$, then either

(i.1) $a \in X$ or $b \in X$ (in which case $L \cap C \neq \emptyset$, because $a, b \in C$), or

(i.2) $X \not\subseteq L_a \cup L_b$ (in which case $L \cap C \neq \emptyset$).

So $L \cap C \neq \emptyset$. If $L \cap L_a \neq \emptyset$, then either

(ii.1) $L \subseteq L_b$ (in which case $L = L_b$, by Lemma 1.1, so $b \in L$, so $L \cap C \neq \emptyset$), or

(ii.2) $L \not\subseteq L_b$ (so that $L \not\subseteq L_a \cup L_b$, so, as before, $L \cap C \neq \emptyset$).

Similarly, if $L \cap L_b = \emptyset$, then $L \cap C \neq \emptyset$. So C is a cs for $\mathcal{L}$.

(2) The set $(C-\{a\}) \cap L_a$ is empty, so $C-\{a\}$ is not a cs for $\mathcal{L}$. The set $(C-\{b\}) \cap L_b$ is empty, so $C - \{b\}$ is not a cs for $\mathcal{L}$.

Therefore $\mathcal{L}$ is not pre-normal.

$\Leftarrow$: Let C be a cs, let $a, b \in L \in \mathcal{L}$. Assume that $C - \{a\}$ is not a cs of $\mathcal{L}$ and that $C - \{b\}$ is not a cs of $\mathcal{L}$.

Since C - {a} is not a cs of $\mathcal{L}$, there is a set L_a in $\mathcal{L}$ such that $(C-\{a\}) \cap L_a = \emptyset$ (but $C \cap L_a \neq \emptyset$). Thus $C \cap L_a = \{a\}$, but since b is in C - {a}, b is not in L_a.

Similarly, there is a set L_b such that $b \in L_b$ but $a \notin L_b$. Thus $\langle a,b,L_a,L_b \rangle$ is a crossed quadruple.

Now $C - \{a,b\} \subseteq C - \{a\}$, and $(C-\{a\}) \cap L_a = \emptyset$, so $(C-\{a,b\}) \cap L_a = \emptyset$. Similarly $(C-\{a,b\}) \cap L_b = \emptyset$. So $(C-\{a,b\}) \cap (L_a \cup L_b) = \emptyset$. So $C \cap [(L_a \cup L_b) - \{a,b\}] = \emptyset$. So there is no set X in L satisfying $X \subseteq [(L_a \cup L_b) - \{a,b\}]$ (since C is a cs for $\mathcal{L}$). Thus the crossed quadruple $\langle a,b,L_a,L_b \rangle$ has no cross set in $\mathcal{L}$. Q.E.D.

Checking for cross sets is usually easier than checking for pre-normality. In fact, in a later section, we will give visual meaning to add to our intuitions about cross sets. Also we will use cross sets to show that the weave $\mathcal{L}$ = {{a,x}, {a,d,b}, {b,y}} is a very important counterexample.

If we assume that $\mathcal{L}$ has cross sets, and try to prove The Fundamental Theorem, we succeed until we reach the point of Lemma 2.10. Without Lemma 2.10 we get "trees" that aren't necessarily well-founded as in the figure on page 65. This leads us to the question "What new kinds of trees and games can we define?" We present an interesting example.

The alphabet of $\mathcal{L}$ is $\mathbb{R}^{\geq 0} \cup \tilde{\mathbb{R}}^{\geq 0}$--two copies of the set of all real numbers greater than or equal to zero. Let $\mathcal{L}$ be all sets of the form

$$\{r\} \cup \{\tilde{s} \mid s < r\},$$

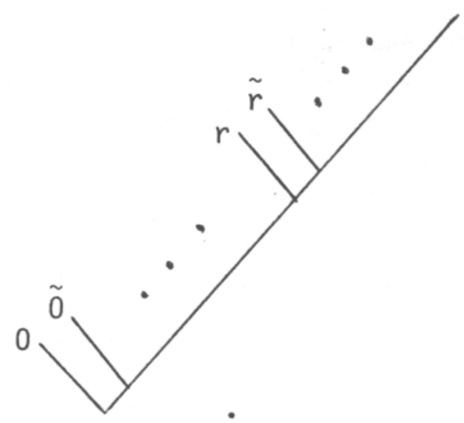
where $r \in \mathbb{R}^{\geq 0}$. Then $\mathcal{R}$ must be all sets of the form

$$\{\tilde{s}\} \cup \{r \mid r \leq s\},$$

where $s \in \mathbb{R}^{\geq 0}$, plus the set

$$\{\tilde{s} \mid s \in \mathbb{R}^{<0}\}.$$

If we try to make a tree to represent this weave, we come up with



Here, for each r in $\mathbb{R}^{\geq 0}$, we have a pair of branches extending from the backbone. In each pair, the node directly below r is taken to be a I-node, and the node directly below $\tilde{r}$ is taken to be a II-node. The set $\{r\} \cup \{\tilde{s} \mid s < r\}$ is represented by the I-strategy in which player I decides to turn off the backbone <u>only</u> at the node directly below r.

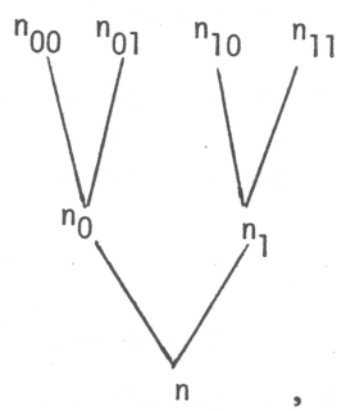
Not only is this tree not well-founded--it doesn't even present a discrete order of play. The problem here is that finding a generalized definition for "tree" and "strategy" seems to be very difficult. We would like to see such notions defined. Also we would like to see a theory of continuing trees, such as that developed in Section 3, also developed for trees of this generalized type.

SECTION 8: A DETERMINATENESS LEMMA

In this section we take a brief detour to look at the determinateness problem from a different point of view.

We begin with a stopped tree $\mathcal{T}$. (For our examples, and for the proof of the lemma, we will use a binary tree, but everything we say holds for any tree with the property that the successors of each node form a well-ordered set.) Let $\hat{\Sigma}_I$ be the set of all $\hat{\sigma}$ such that σ is a I-strategy on $\mathcal{T}$, and let $\hat{\Sigma}_{II}$ be the set of all $\hat{\tau}$ such that τ is a II-strategy on $\mathcal{T}$. The pair $<\hat{\Sigma}_I, \hat{\Sigma}_{II}>$ is a decomposable weave, and, as usual, $\hat{\Sigma}_I = \mathcal{G}(\hat{\Sigma}_{II})$ and $\hat{\Sigma}_{II} = \mathcal{G}(\hat{\Sigma}_I)$. A set X of branches on $\mathcal{T}$ is called <u>indeterminate</u> iff there is no I-strategy σ such that $\hat{\sigma} \subseteq X$, and there is no II-strategy τ such that $\hat{\tau} \subseteq \hat{\mathcal{T}} - X$. Rephrasing this in the terminology of weaves, we get that X is indeterminate iff X is a cs of $\hat{\Sigma}_I$, and there is no $X' \subseteq X$ such that X' is a gcs of $\hat{\Sigma}_I$.

What does a cs of $\hat{\Sigma}_I$ look like on the tree? First of all it is a subset of the set of all branches. Say, for example, the tree begins with



,

and we want to form a cs X on it. At least one branch of X must pass through either n_0 or n_1 (otherwise $X = \emptyset$). Furthermore, there must be branches b_0 and b_1 in X such that

(1) b_0 passes through n_{00}, and b_1 passes through n_{01}, or

(2) b_0 passes through n_{10}, and b_1 passes through n_{11}.

This follows from the fact that X is a cs of $\hat{\Sigma}_I$. We can continue in this way to identify sets of nodes that seem to be forming a II-strategy.

The question is, "What can go wrong?" In the finite levels of the construction we are forming a set that has great promise of becoming a II-strategy (with possibly other branches added). After taking away the other branches, we should get X', a II-strategy, and thus a gcs of $\hat{\Sigma}_I$. It looks as if we're proving that, for every cs X, we can find a gcs $X' \subseteq X$. But since indeterminate sets exist, the set that we're forming must, in some instances, be less than a II-strategy. There are two things that might possibly be going wrong.

(1) In the construction of X, some branch b <u>seems</u> to be going into X because all the nodes on b are becoming part of X (i.e., are nodes on branches that <u>are</u> in X) but, in fact, b is not in X. This is the case with the binary tree and the set X_0, where X_0 is the set of all sequences of zeros and ones except the sequence <0,0,0,...>. The branch <0,0,0,...> "seems" to be in X_0, because all of its nodes appear on branches that are in X_0, but actually <0,0,0,...> is not in X_0.

More precisely, let Y be any set of branches on $\mathcal{T}$. Let Y* be the set of all branches b on $\mathcal{T}$ such that every node of b appears on some branch b' of

Y. The set Y* is called the completion of Y. It could be that, in forming a cs X, we will find that X* contains a II-strategy but that X does not. (Notice that in this construction we only said which nodes had branches b_0, b_1, of X passing through them, i.e., for which nodes n_{00}, n_{01} there exist branches b_0 and b_1 of X such that n_{00} is on b_0 and n_{01} is on b_1.)

(2) The other thing that might possibly go wrong is that, although X* (if not X) has, at each finite stage in the construction, a subset which is a II-strategy for the tree up to that stage, it might not have such a subset after infinitely many stages (i.e., when the construction of X is finished).

The purpose of the following lemma is to show that (2) cannot happen. The set X* will always have a subset which is a II-strategy. This means that if X is an indeterminate set, it can only be because there are branches b in X such that every node of b is on some branch b' that is in X, but b itself is not in X. Looking at it from this point of view, it becomes clear why "open sets" is the first proposal in trying to guarantee the determinateness of a set. It is hoped that this point of view will generate other directions besides the Borel hierarchy for guaranteeing determinateness. For instance, these "bad" branches that seem to be in X (but really are not) are analoguous to inadmissible strategies on continuing trees.

This notion of the completion of a set is used in the next section to attack another problem.

LEMMA 8.1: Let $\mathcal{T}$ be a stopped binary tree. Let $\hat{\Sigma}_I$ be the set of all $\hat{\sigma}$ such that σ is a I-strategy on $\mathcal{T}$. Let X be a cs for $\hat{\Sigma}_I$. Then there is a subset Y of X such that $Y^* = \hat{\tau}$ for some II-strategy τ.

PROOF: If Z is a set of branches, and n is a node, we will say that n is on Z iff there is a branch b such that n is on b and b is in Z.

Let $\mathcal{T}_i$ be the tree of all nodes of $\mathcal{T}$ of level less than or equal to i. Similarly define H_i, for any set H of nodes on $\mathcal{T}$, and S_i, for any set S of branches on $\mathcal{T}$.

The set X_i is a cs for the set of I-strategies on $\mathcal{T}_i$. Since $\mathcal{T}_i$ is finite, there is a subset A_i of X_i such that A_i is a gcs for the set of I-strategies on $\mathcal{T}_i$. So there is a II-imposed subgame of $\mathcal{T}_i$ which is a subset of X_i. Let A_i and B_i be two such II-imposed subgames. Then $A_i \cup B_i$ is a II-imposed subgame of $\mathcal{T}_i$, and $A_i \cup B_i \subseteq X_i$. (This is because any I-node n on $A_i \cup B_i$ is either on A_i or on B_i, so all successors of n are on A_i or on B_i, so all successors of n are on $A_i \cup B_i$. For any node m on $A_i \cup B_i$, all predecessors of m are in $A_i \cup B_i$. Any branch of $A_i \cup B_i$ goes all the way up to level i.) So there is a <u>largest</u> II-imposed subgame of $\mathcal{T}_i$ which is a subset of X_i. Call it G_i.

If $i \geq j$, let $(G_i)_j$ be all nodes of G_i of level less than or equal to j. Notice that $(G_i)_j$ is a II-imposed subgame of $\mathcal{T}_j$ and $(G_i)_j \subseteq X_j$. So $(G_i)_j \subseteq G_j$. This implies that $((G_i)_j)_k \subseteq (G_j)_k$ if $k \leq j \leq i$. Notice also that $((G_i)_j)_k = (G_i)_k$. Therefore $(G_i)_k \subseteq (G_j)_k$.

Thus if we fix k, we find that $(G_i)_k$ is a II-imposed subgame of $\mathcal{T}_k$, for each i, and these sets are getting smaller as i increases. So there must be a number $I \geq k$ such that, for all $i \geq I$, $(G_i)_k = (G_{i+1})_k$. (This is where the well-foundedness assumption, mentioned in the beginning of this section, is used. There are only finitely many nodes that we can remove from G_k and still have a II-strategy for $\mathcal{T}_k$.)

So, for each k, there is a number I_k such that, for all $i \geq I_k$, $(G_i)_k = (G_{i+1})_k$. Let Y be $\bigcup_k (G_{I_k})_k$. It can be checked that Y is a subset of X, and for some II-strategy τ of $\mathcal{T}$, $Y \subseteq \hat{\tau}$. Q.E.D.

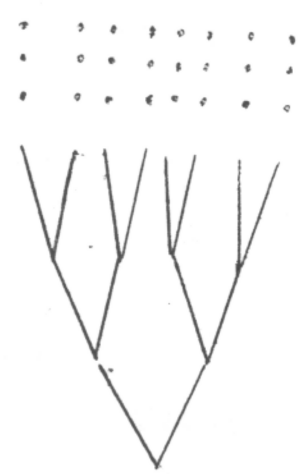
SECTION 9: THE NORMALITY PROBLEM

We would like to explore the relationship between "being normal" and "being a tree". The earlier sections dealt with weaves that we assumed were normal and showed how they can be made into trees. Now we start with weaves that we know are trees and ask "When are such weaves normal?" We will say that a tree $\mathcal{T}$ is <u>normal</u> iff the weave corresponding to $\mathcal{T}$ is normal.

We already know some things about this problem.

(1) From The Fundamental Theorem we know that, in order for a tree to be normal, it must be a well-founded tree (not a generalized tree of the sort discussed at the end of Section 7). From now on in this section we will discuss only well-founded trees.

(2) Let $\mathcal{K}$ be the following binary tree with final nodes.

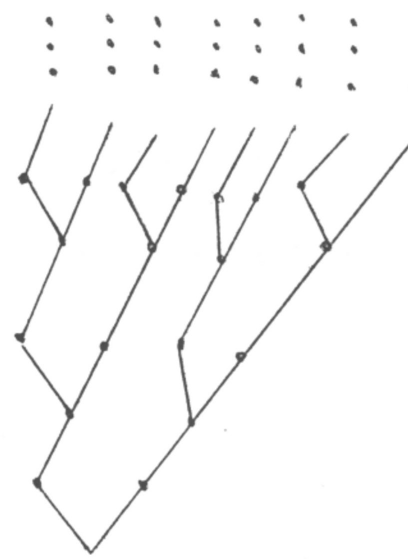


Then a well-known determinateness result [3] tells us that $\mathcal{K}$ is not normal.

(3) Let $\mathcal{T}$ be a tree, and assume that every branch of $\mathcal{T}$ is finite. Then every game that can be played on $\mathcal{T}$ is an open game and thus a determinated game, and so $\mathcal{T}$ is normal.

(4) Any stopped tree with countably many branches is normal, because, in a countable topological space, every set is an F_σ set, and it has been shown in [8] that F_σ sets are determined.

(5) The following tree



(with a final node at the end of each branch) has $2^{\aleph_0}$ branches, but is normal.

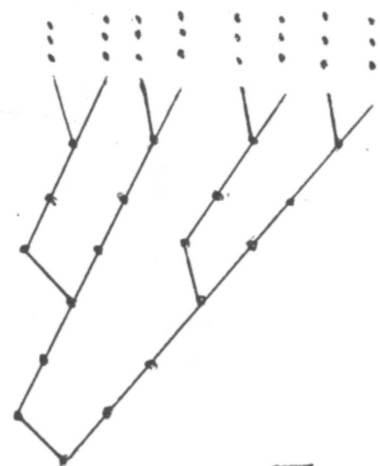
(6) The tree

← $2^{\aleph_0}$--many branches each of length 1

has $2^{\aleph_0}$ branches, but it is normal.

(7) Let $\mathcal{T}$ be a tree such that the first ω levels of $\mathcal{T}$ are identical to $\mathcal{K}$. Then $\mathcal{T}$ is "bigger" than $\mathcal{K}$ in some sense, and so $\mathcal{T}$ is non-normal.

(8) Let $\mathcal{T}$ be the following tree.



Then $\mathcal{T}$ is "bigger" than $\mathcal{K}$ in some sense, and $\mathcal{T}$ is non-normal.

We would like to generalize the statements of (6), (7), and (8). For (6) we would like to compare the given tree with the binary tree

in a way that can be generalized to longer trees, and show that such trees are always normal. In other words, we would like a theorem to show that the width of a tree doesn't help contribute to the possibility of its being non-normal. So far, no such theorem has appeared. We suggest that a possible avenue might be found by looking at Section 6 of [6].

For (7) and (8) we will clarify the notion of "bigger" and show that a tree that's "bigger" than a non-normal tree is itself non-normal. This means that, somewhere in the partial ordering of binary trees (with respect to "bigness"), there is a dividing line between normal and non-normal. So far we know that $\mathcal{K}$ is on the non-normal side of this dividing line, but we don't know if any trees smaller than $\mathcal{K}$ are non-normal. For the most part we will discuss only binary trees.

The first task is to get a definition of "containment" for trees.

DEFINITION (for stopped trees $\mathcal{T}_1$ and $\mathcal{T}_2$): The tree $\mathcal{T}_2$ is an <u>extension</u> of $\mathcal{T}_1$ iff $\mathcal{T}_1$ is a subtree of $\mathcal{T}_2$.

THEOREM 9.1 (for stopped trees): An extension of a non-normal tree is non-normal.

PROOF: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be stopped trees, and let $\mathcal{T}_2$ be an extension of $\mathcal{T}_1$. Let X_1 be a set of branches witnessing the non-normality of $\mathcal{T}_1$ (i.e., for each σ which is a I-strategy on $\mathcal{T}_1$, $\hat{\sigma} \nsubseteq X_1$, and, for each τ which is a II-strategy on $\mathcal{T}_1$, $\hat{\tau} \cap X_1 = \emptyset$). Let $X_2 = \{b_2 \in \hat{\mathcal{T}}_2 \mid \exists\, b_1 \in X_1$ s.t. b_2 extends $b_1\}$. Notice that $X_1 = \{b_1 \in \hat{\mathcal{T}}_1 \mid \exists\, b_2 \in \hat{\mathcal{T}}_2$ s.t. $(b_2$ extends $b_1) \wedge (b_2 \in X_2)\}$. So

$$S \subseteq X_1 \text{ iff } \forall s \in S \ \exists b_2 \in \hat{\mathcal{T}}_2 \text{ s.t. } b_2 \text{ extends } b_1 \text{ and } b_2 \in X_2. \qquad (*)$$

We can also write $X_1 = \{b_1 \in \hat{\mathcal{T}}_1 \mid (\forall b_2 \in \hat{\mathcal{T}}_2)\,[b_2 \text{ extends } b_1 \Rightarrow b_2 \in X_2]\}$. So

$$S \cap X_1 = \emptyset \text{ iff } \forall s \in S \ \ \exists b_2 \in \hat{\mathcal{T}}_2 \text{ s.t. } b_2 \text{ extends } b_1$$
$$\text{and } b_2 \notin X_2. \qquad (**)$$

Let $\sigma(\tau)$ be a I(II)-strategy on $\hat{\mathcal{T}}_2$. Then the restriction of $\sigma(\tau)$ to $\mathcal{T}_1$ is a I(II)-strategy on $\mathcal{T}_1$. We show that the set X_2 witnesses the non-normality of $\mathcal{T}_2$.

(i) If σ_2 is a I-strategy on $\mathcal{T}_2$, and $\hat{\sigma}_2 \subseteq X_2$, let σ_1 be the restriction of σ_2 to $\mathcal{T}_1$. Then σ_1 is a I-strategy on $\mathcal{T}_1$, and, by (*), $\hat{\sigma}_1 \subseteq X_1$. This is a contradiction.

(ii) If τ_2 is a II-strategy on $\mathcal{T}_2$, and $\hat{\tau}_2 \cap X_2 = \emptyset$, let τ_1 be the restriction of τ_2 to $\mathcal{T}_1$. Then τ_1 is a II-strategy in $\mathcal{T}_1$, and, by (**), $\hat{\tau}_1 \cap X_1 = \emptyset$. This is a contradiction.

Therefore $\mathcal{T}_2$ is non-normal. Q.E.D.

DEFINITION (for continuing trees $\mathcal{T}_1$ and $\mathcal{T}_2$): The tree $\mathcal{T}_2$ is an <u>extension</u> of $\mathcal{T}_1$ iff $\mathcal{T}_1$ is a subtree of $\mathcal{T}_2$, and no branch of $\mathcal{T}_2$ is a proper extension of a continuing branch of $\mathcal{T}_1$.

THEOREM 9.2 (for continuing trees): An extension of a non-normal tree is non-normal.

The proof is similar to the proof of Theorem 9.1. As usual we must be careful to discuss only stopped branches and only admissible strategies.

DEFINITION: Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be trees, and let f be a function from $\hat{\mathcal{T}}_1$ to $\hat{\mathcal{T}}_2$. The function f is an <u>isomorphism</u> iff it satisfies the following conditions.

(1) The function is one-to-one and onto.

(2) If b is a stopped (continuing) branch, then f(b) is a stopped (continuing) branch.

(3) If b_1 and b_2 part company at an even (odd) node, then $f(b_1)$ and $f(b_2)$ part company at an even (odd) node.

By this definition, the tree in example (8) is isomorphic to $\mathcal{K}$.

DEFINITION: The tree $\mathcal{T}_1$ is <u>smaller than</u> $\mathcal{T}_2$ (and $\mathcal{T}_2$ is <u>larger than</u> $\mathcal{T}_1$) iff there is a tree $\mathcal{T}_1'$ satisfying

(1) $\mathcal{T}_1'$ is isomorphic to $\mathcal{T}_1$, and

(2) $\mathcal{T}_2$ is an extension of $\mathcal{T}_1'$.

THEOREM 9.3: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic, then they are either both normal or both non-normal.

This theorem will be proved in the next section.

COROLLARY: If $\mathcal{T}_1$ is smaller than $\mathcal{T}_2$, and $\mathcal{T}_1$ is non-normal, then $\mathcal{T}_2$ is non-normal.

Thus any tree that's larger than $\mathcal{K}$ is non-normal. What about trees smaller than $\mathcal{K}$? We know that we can add normal "pieces" to normal trees by the next theorem.

DEFINITION: If $\mathcal{T}$ is a tree and x is a node on $\mathcal{T}$, then $\mathcal{T}_x$ is the subtree of $\mathcal{T}$ consisting of all branches of $\mathcal{T}$ that pass through x.

THEOREM 9.4: Let $\mathcal{T}$ be a stopped tree and $\mathcal{T}_{x_0}$, $\mathcal{T}_{x_1}$, ..., $\mathcal{T}_{x_\alpha}$, be subtrees. (For each i, let the level of the node x_i be a successor ordinal.) Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by removing the subtrees $\mathcal{T}_{x_0}$, $\mathcal{T}_{x_1}$, ..., $\mathcal{T}_{x_\alpha}$, ... If $\mathcal{T}'$ is normal, then $\mathcal{T}$ is normal.

PROOF: Let X be a set of branches of $\mathcal{T}$, and consider X to be the winning set for player I. We will show that this game $< \mathcal{T}, X>$ is determined.

Define a set, Y', of nodes on $\mathcal{T}'$ as follows. Let $y \in Y'$ iff

(1) y is a I-node,

(2) y is x_α, for some α,

(3) player I has a winning strategy for the game $< \mathcal{T}_{x_\alpha}, X \cap \hat{\mathcal{T}}_{x_\alpha}>$,

and

(4) there is no node, n, below y, satisfying

(4a) n is a II-node,

(4b) n is x_β, for some β, and

(4c) player II has a winning strategy for the game $< \mathcal{T}_{x_\beta}, X \cap \hat{\mathcal{T}}_{x_\beta}>$.

Let $\Gamma_{Y'}$ be the set of all branches that pass through a node y in Y'. Let Z' be the set of all branches z in $\hat{\mathcal{T}}' \cap X$ satisfying

"there is no node n, along z, with properties (4a), (4b) and (4c)".

The tree $\mathcal{T}'$ is normal, so either player I or player II has a winning strategy for the game $< \mathcal{T}', \Gamma_{Y'} \cup Z'>$.

Case(1) Player I has a winning strategy in $<\mathcal{T}', \Gamma_{Y'} \cup Z'>$. Then player I can win $<\mathcal{T}, X>$ in the following way. She should first play her strategy for $<\mathcal{T}', \Gamma_{Y'} \cup Z'>$. In doing so, any of three things may happen.

Subcase(1.1) The game remains in $\mathcal{T}'$. Then the resulting play will be in Z'. Since $Z' \subseteq X$, player I will have won $<\mathcal{T}, X>$.

Subcase(1.2) Player I moves the game off of $\mathcal{T}'$. Then the game will have reached a node y in Y'. By definition of Y', player I has a winning strategy in the game $<\mathcal{T}_y, X \cap \hat{\mathcal{T}}_y>$ (property (3)), so player I should proceed to win the game $<\mathcal{T}_y, X \cap \hat{\mathcal{T}}_y>$, and thus win $<\mathcal{T}, X>$.

Subcase(1.3) Player II moves the game off of $\mathcal{T}'$. Then the game will have reached a node x_α (for some α). Since the game has been played (up to that point) only along branches in $Z' \cap \Gamma_{Y'}$, it is impossible for the node x_α to satisfy the conditions (4a), (4b) and (4c). Thus, in particular, player II cannot have a winning strategy in the subgame $<\mathcal{T}_\alpha, \hat{\mathcal{T}}_\alpha \cap X>$. Thus player I has a winning strategy in $<\mathcal{T}_\alpha, \hat{\mathcal{T}}_\alpha \cap X>$, and so player I can proceed to win $<\mathcal{T}_\alpha, \hat{\mathcal{T}}_\alpha \cap X>$, and thus win $<\mathcal{T}, X>$.

Therefore, in Case(1), player I has a winning strategy for $<\mathcal{T}, X>$.

Case (2) Player II has a winning strategy in $<\mathcal{T}', \Gamma_{Y'} \cup Z'>$. Then player II has a winning strategy in $<\mathcal{T}, X>$. The proof of this is similar to the proof for Case(1). Q.E.D.

COROLLARY: If $\mathcal{T}'$ is normal and stopped, then any extension of $\mathcal{T}'$ that is obtained by adding only countably many branches is also normal.

COROLLARY: If $\mathcal{T}'$ is normal and stopped, then any extension of $\mathcal{T}'$ that is obtained by adding only nodes of finite level is also normal.

COROLLARY: If $\mathcal{T}$ is non-normal and stopped, and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by removing all nodes n such that there is no node n', of infinite level, above n, then $\mathcal{T}'$ is non-normal.

COROLLARY: If $\mathcal{T}$ is non-normal and stopped, and $\mathcal{T}'$ is obtained from $\mathcal{T}$ by removing all finite branches, then $\mathcal{T}'$ is non-normal.

We want to be able to go from stopped trees to continuing trees. The following definitions and theorem help.

DEFINITION: Let $\mathcal{T}$ be a continuing tree. The <u>stopping</u> of $\mathcal{T}$ is the tree that is obtained from $\mathcal{T}$ by replacing

by

for each continuing branch b.

DEFINITION: If $\mathcal{T}'$ is the stopping of $\mathcal{T}$, then $\mathcal{T}$ is called <u>an unstopping</u> of $\mathcal{T}'$.

THEOREM 9.5: If $\mathcal{T}$ is an unstopping of $\mathcal{T}'$, and $\mathcal{T}'$ is normal, then $\mathcal{T}$ is normal.

PROOF: Let $X \subseteq \hat{\mathcal{T}}$. Let $X' = X \cup \{b \in \hat{\mathcal{T}}' \mid$ b is continuing (as a branch of $\mathcal{T}$) and b has the marker "$\exists$ L, $\neg \exists$ R"$\}$. By the normality of $\mathcal{T}'$, there is a I-strategy, σ', of $\mathcal{T}'$, such that $\hat{\sigma}' \subseteq X'$, or there is a II-strategy, τ', of $\mathcal{T}'$, such that $\hat{\tau}' \subseteq \hat{\mathcal{T}}' - X'$.

Case(1) Let $\hat{\sigma}' \subseteq X'$. Notice that σ' passes through no branches that are marked "$\neg \exists$ L, $\exists$ R" in $\mathcal{T}$, so σ' is an admissible I-strategy in $\mathcal{T}$. When interpreted as a strategy in $\mathcal{T}$, $\hat{\sigma}' \subseteq X$.

Case(2) Let $\hat{\tau}' \subseteq \hat{\mathcal{T}}' - X'$. Since X' contains all branches with the marker "$\exists$ L, $\neg \exists$ R", τ' passes through none of these branches. So τ' is an admissible II-strategy in $\mathcal{T}$. Since $\hat{\tau}' \cap X' = \emptyset$, we have $\hat{\tau}' \cap X = \emptyset$. So $\hat{\tau}' \subseteq \hat{\mathcal{T}} - X$, when interpreted as a strategy in $\mathcal{T}$.

Thus the game $< \mathcal{T}, X>$ is determined. So $\mathcal{T}$ is normal. Q.E.D.

The question remains "Are there any stopped trees of length at most $\omega + 1$, other than $\mathcal{K}$, which are non-normal?" In an attempt to answer this we have formulated the following machinery.

LEMMA 9.6: Let $X \subseteq \hat{\mathcal{T}}$, $\mathcal{T}$ a I-imposed subtree of $\mathcal{K}$, and let the game $< \mathcal{K}, X>$ be indeterminate. Then $< \mathcal{T}, X>$ is indeterminate.

PROOF: Assume that the game $<\mathcal{T}, X>$ is determined.

Case(1) Player I has a winning strategy for $<\mathcal{T}, X>$. Then to win $<\mathcal{R}, X>$, player I should use her strategy for $<\mathcal{T}, X>$, making sure not to leave the tree $\mathcal{T}$. (She can do this since $\mathcal{T}$ is I-imposed.)

Case(2) Player II has a winning strategy for $<\mathcal{T}, X>$. Then to win $<\mathcal{R}, X>$, player II should use her strategy for $<\mathcal{T}, X>$ until the resulting play leaves the tree $\mathcal{T}$. (At that time player II is sure to win, since $X \subseteq \hat{\mathcal{T}}$.) If the resulting play remains on $\mathcal{T}$, then player II wins $<\mathcal{T}, X>$, and thus wins $<\mathcal{R}, X>$.

Therefore, if $<\mathcal{T}, X>$ is determined then $<\mathcal{R}, X>$ is determined. Q.E.D.

Suppose we can find such an X and $\mathcal{T}$. If we can show that $\mathcal{R}$ is not smaller than $\mathcal{T}$, then $\mathcal{T}$ is an example of a non-normal tree which is strictly smaller than $\mathcal{R}$. The converse of this lemma also holds.

LEMMA 9.7: Let $\mathcal{T}$ be a I-imposed subtree of $\mathcal{R}$, and let $X \subseteq \hat{\mathcal{T}}$. If $<\mathcal{T}, X>$ is indeterminate, then $<\mathcal{R}, X>$ is indeterminate.

PROOF: Assume that $<\mathcal{R}, X>$ is determined.

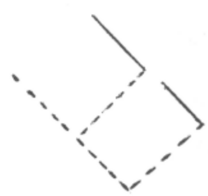
Case (1) Player I has a winning strategy for $<\mathcal{R}, X>$. Then player I should use her winning strategy for $<\mathcal{R}, X>$. Since $X \subseteq \hat{\mathcal{T}}$, the resulting play (of the game $<\mathcal{R}, X>$) will be on the tree $\mathcal{T}$, thus being a win for I in $<\mathcal{T}, X>$.

Case (2) Player II has a winning strategy for $<\mathcal{R}, X>$. Then player II should use her winning strategy for $<\mathcal{R}, X>$. In this way, she will win $<\mathcal{T}, X>$.

Therefore, if $<\mathcal{R}, X>$ is determined, then $<\mathcal{T}, X>$ is determined. Q.E.D.

The next step is to show how we can get $\mathcal{T}$ from X. Given X, we will find a smallest possible I-imposed subtree $\mathcal{T}$ such that $X \subseteq \hat{\mathcal{T}}$.

Recall from Section 8 the definition of the completion, X*, of a set X of branches. If $X \subseteq \hat{\mathcal{T}}$, then $X^* \subseteq \hat{\mathcal{T}}$. Let $\mathcal{S}$ be the subtree of $\mathcal{K}$ satisfying $\hat{\mathcal{S}} = X^*$. It may be that $\mathcal{S}$ is not a I-imposed subtree of $\mathcal{K}$. That is, there may be a node n on $\mathcal{S}$ of II-parity, and a node x of $\mathcal{K}$, such that x is a successor of n in $\mathcal{K}$, but x is not in $\mathcal{S}$. For each such n and x, enlarge $\mathcal{S}$ by adding a certain subtree $\mathcal{S}_x$ to $\mathcal{S}$ --let $\mathcal{S}_x$ be a I-strategy starting from x. For example, the tree



becomes enlarged to be the tree

The enlarged set is our I-imposed set $\mathcal{T}$.

LEMMA 9.8: Construct $\mathcal{T}$ from X as above. Then $\mathcal{T}$ is the smallest I-imposed subtree of $\mathcal{K}$ satisfying $X \subseteq \hat{\mathcal{T}}$.

Begin with $<\mathcal{K}, X>$ indeterminate, and construct $\mathcal{T}$ from X. By Lemma 9.6, X must be indeterminate in $\mathcal{T}$. Now we can get another converse to Lemma 9.6.

LEMMA 9.9: Let $\mathcal{T}$ be a non-normal tree which is strictly smaller than $\mathcal{K}$. Then there is a set $X \subseteq \hat{\mathcal{K}}$, and a I-imposed subtree $\mathcal{T}'$ of $\mathcal{K}$, such that $X \subseteq \hat{\mathcal{T}}'$.

PROOF: Enlarge $\mathcal{T}$ into a I-imposed subtree $\mathcal{T}'$ (of $\mathcal{K}$) as in the procedure given above. Let X be a set of branches witnessing the non-normality of $\mathcal{T}$. By Lemma 9.8, $\mathcal{T}'$ is a I-imposed subtree of $\mathcal{K}$. By Lemma 9.6, $\mathcal{T}'$ is not normal. Since the subgames that we added to $\mathcal{T}$ to obtain $\mathcal{T}'$ were I-strategies, $\mathcal{T}'$ is still strictly smaller than $\mathcal{K}$. Q.E.D.

Thus to find a non-normal tree that's strictly smaller than $\mathcal{K}$, we should look for a I-imposed subtree, $\mathcal{T}$, of $\mathcal{K}$ and a set $X \subseteq \hat{\mathcal{T}}$ witnessing the non-normality of $\mathcal{K}$. By the construction of $\mathcal{T}$ from X we need only look for sets X satisfying

(1) X witnesses the non-normality of $\mathcal{K}$, and

(2) X^* is strictly smaller than $\mathcal{K}$.

Whether or not such a set X exists is an open question. One possible approach is to try to construct X by modifying the technique used in [3].

SECTION 10: GRAPHS

In previous sections we have represented weaves by trees and by logical statements. In this section we represent weaves by graphs. The mapping from weaves to graphs is not onto. Graphs that lie in the range of this mapping will be called "coverable". The mapping from weaves to coverable graphs is not one-to-one, but it has a fairly natural partial inverse. The weaves that are in the range of this partial inverse will be called "graphable", and so we will have a one-to-one function from graphable weaves to coverable graphs. Although the theory of graphable weaves is not developed fully here, we believe that this is a valuable approach, since it shifts the focus of the subject in an interesting way and provides intuitions in that direction.

DEFINITION: Let $\mathcal{L}$ be a weave, and let a and b be in Alph($\mathcal{L}$). We write $a \approx_{\mathcal{L}} b$ iff there is a set L in $\mathcal{L}$ such that $a, b \in L$.

PROPOSITION 10.1: The relation $\approx_{\mathcal{L}}$ is reflexive and symmetric.

For any weave $\mathcal{L}$, we can use $\approx_{\mathcal{L}}$ to form an undirected graph. We call this the <u>graph</u> of $\mathcal{L}$ and denote it by graph($\mathcal{L}$).

DEFINITION: Let ~ be a reflexive, symmetric relation. A set S is called a <u>clique</u> iff, for all a and b, if $a, b \in S$, then $a \sim b$. The clique S is a <u>maximal clique</u> iff, for each $c \notin S$, there is an $a \in S$ such that $a \nsim c$.

For any weave $\mathcal{L}$, and any set L in $\mathcal{L}$, L is a clique of $\approx_{\mathcal{L}}$. For some weaves the sets correspond exactly to the maximal cliques of $\mathcal{L}$, and we call these weaves <u>graphable</u>.

DEFINITION: Let $\sim$ be a reflexive, symmetric relation. The family of maximal cliques of $\sim$ is denoted $\mathcal{L}(\sim)$.

So a weave $\mathcal{L}$ is graphable if and only if $\mathcal{L}(\approx_{\mathcal{L}}) = \mathcal{L}$.

PROPOSITION 10.2: The Axiom of Choice implies that $\approx_{\mathcal{L}(\sim)}$ is equal to $\sim$, for any relation $\sim$, which is reflexive and symmetric.

PROOF: For any a, b ε Alph($\mathcal{L}$), $a \approx_{\mathcal{L}} b$ iff there is a set S in $\mathcal{L}(\sim)$ such that a, b ε S. This, in turn, is true iff there is a set S such that

(1) for each x and y, if x, y ε S then $x \sim y$,

(2) S is maximal with respect to property (1), and

(3) a, b ε S.

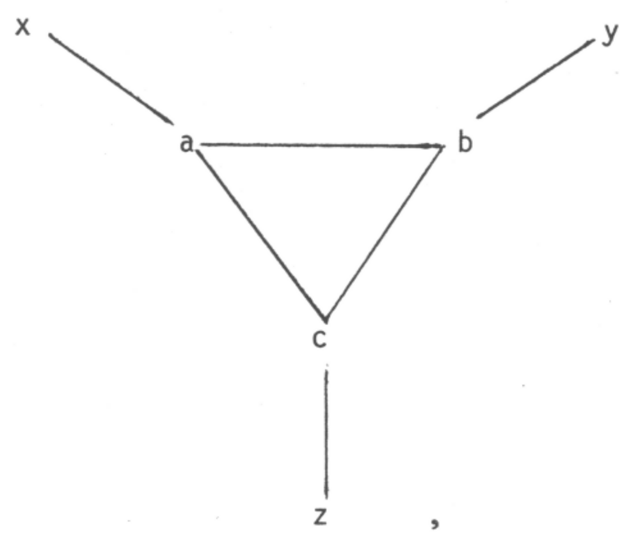
We want to show that $a \approx_{\mathcal{L}(\sim)} b$ iff $a \sim b$.

$\Longrightarrow$: If $a \approx_{\mathcal{L}(\sim)} b$, then a, b ε S, and for each x and y, if x, y ε S, then $x \sim y$. So $a \sim b$.

$\Longleftarrow$: Assume that $a \sim b$. Let S ε $\mathcal{S}$ iff for all x and y, if x, y ε S, then $x \sim y$. Order the family $\mathcal{S}$ by set inclusion. This family is non-empty, because $\{a,b\} \in \mathcal{S}$. Let $S_i \in \mathcal{S}$, for all $i < \alpha$, and let $S_\beta \subseteq S_\gamma$ for $\beta < \gamma < \alpha$. We want to show that $\bigcup S_i \in \mathcal{S}$. If x, y ε $\bigcup S_i$, then let $x \in S_\beta$, $y \in S_\gamma$, and $S_\beta \subseteq S_\gamma$. Then x, y ε S_γ, so $x \sim y$. Therefore $\bigcup S_i \in \mathcal{S}$.

Thus $\mathcal{S}$ has a maximal element S*, so there is a set S* in $\mathcal{L}(\sim)$ such that a, b ε S*. Therefore a $\approx_{\mathcal{L}(\sim)}$b. Q.E.D.

Let $\mathcal{L}$ be the weave {{a,b,x}, {b,c,y}, {a,c,z}}. The graph of $\mathcal{L}$ is



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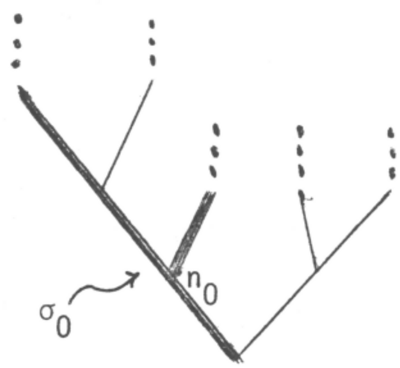
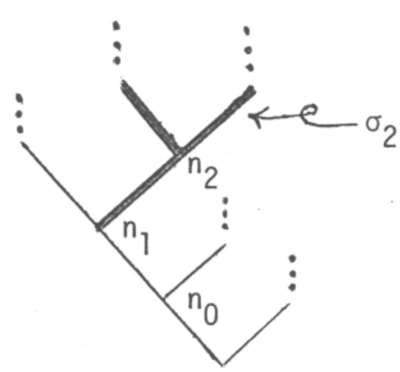
and so $\mathcal{L}(\approx_{\mathcal{L}})$ is {{a,b,c}, {x,a}, {b,y}, {c,z}}. In this case, $\mathcal{L}(\approx_{\mathcal{L}})$ is nothing like the original weave $\mathcal{L}$. This is not the case for decomposable weaves.

PROPOSITION 10.3: Let b_0, b_1, ..., b_α, ... be branches on the stopped tree $\mathcal{T}$. Assume that for each i, j < α, b_i and b_j part company at a II-node. Then there is a I-strategy, σ, on $\mathcal{T}$ such that b_0, b_1, ..., b_α, ... are all in $\hat{\sigma}$. (If $\mathcal{T}$ is continuing, we further assume that b_0, b_1, ..., b_α, ... are stopped branches, and show that σ is an admissible I-strategy.)

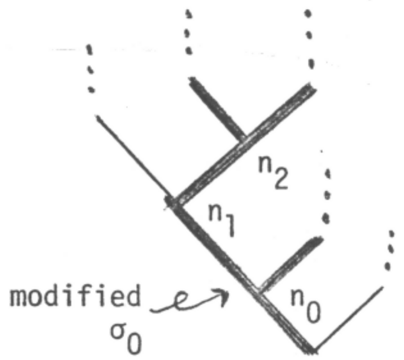
PROOF: First we assume that $\mathcal{T}$ is stopped. There is a unique node, n_0, of lowest level such that two branches, b_{i_0} and b_{j_0}, part company at n_0. Let σ_0 be a I-strategy passing through n_0.



Since n_0 is a node of II-parity, σ_0 passes through each successor, n_1, of n_0. For each such n_1, find the unique node, n_2, of lowest level above n_1 such that two branches b_{i_2} and b_{j_2} part company at n_2. Find a partial I-strategy starting from n_1 and passing through n_2, and call this partial I-strategy σ_2.



Modify σ_0 by trimming away the part of it that follows n_1 and replacing this part by σ_2.



Do this for all successors, n_1, of n_0. For each node n_2 found in this fashion, take each successor n_3, and continue on as above. If we reach a node n_k such that there is no node n_{k+1} above n_k where two branches, b_{i_k} and b_{j_k}, part company, this means that only one branch, b_{i_k}, passes through n_k. Starting from the node n_k, modify σ_0 so that it passes through the branch b_{i_k}. The resulting strategy σ_0 has the property that b_0, b_1, ..., b_α, ... are all in $\hat{\sigma}_0$.

If $\mathcal{T}$ is continuing, we need only be sure that the original strategy that is chosen, σ_0, and all partial strategies that are chosen thereafter (e.g., σ_2), are admissible I-strategies. Such strategies must exist because every node on the tree must have an admissible I-strategy passing through it. (Otherwise no stopped branch passing through that node would be in the alphabet of $\mathcal{L}$.) Q.E.D.

COROLLARY: Every decomposable weave is graphable.

PROOF: Let $\mathcal{L}$ be a decomposable weave and let $L \in \mathcal{L}$. Then L is a clique of $\approx_{\mathcal{L}}$.

To show that L is a maximal clique, consider $\mathcal{T}$, the tree for $\mathcal{L}$. On that tree L is a I-strategy, and thus L is a I-imposed subgame. Let b be a branch that is not in L, and let n be the node of highest level that is on both b and some branch, a, of L. Since L is I-imposed, n must be a node of I-parity. So a and b part company at a node of I-parity. So $a \not\approx_{\mathcal{L}} b$. Thus L is a maximal clique of $\approx_{\mathcal{L}}$.

Conversely, let L be a maximal clique of $\approx_{\mathcal{L}}$. By Proposition 10.3, there is an admissible I-strategy σ on the tree for $\mathcal{L}$ such that $L \subseteq \hat{\sigma}$. By the maximality of L, $L = \hat{\sigma}$. So $L \in \mathcal{L}$. Q.E.D.

Now we can prove Theorem 9.3.

THEOREM 9.3: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are isomorphic, then they are either both normal or both non-normal.

PROOF: Let $\mathcal{T}_1$ be the tree for $\mathcal{L}_1$, and let $\mathcal{T}_2$ be the tree for $\mathcal{L}_2$. We have a one-to-one function, f, from $\hat{\mathcal{T}}_1$ onto $\hat{\mathcal{T}}_2$. This is the same as a one-to-one function, f, from Alph($\mathcal{L}_1$) to Alph($\mathcal{L}_2$). By condition (3) in the definition of an isomorphism, we have that $a \approx_{\mathcal{L}_1} b$ iff $f(a) \approx_{\mathcal{L}_2} f(b)$, for all $a, b \in$ Alph($\mathcal{L}_1$). So $\mathcal{L}_1$ and $\mathcal{L}_2$ have isomorphic graphs. But $\mathcal{L}_1$ and $\mathcal{L}_2$ are decomposable, so by above corollary they are graphable. Thus $\mathcal{L}_1$ equals the family of maximal cliques of graph($\mathcal{L}_1$), which is isomorphic to the family of maximal cliques of graph($\mathcal{L}_2$), which, in turn, is equal to $\mathcal{L}_2$. Therefore $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic as weaves. So they are both normal or both non-normal. Thus the trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are either both normal or both non-normal. Q.E.D.

Now we examine the properties of "being a weave", "being a normal weave", etc., in terms of graphs.

Let $\sim$ be a reflexive, symmetric relation. The graph determined by $\sim$ is said to be coverable iff $\mathcal{L}(\sim)$ is a weave. At the present time there is no good characterization of coverable graphs. Some work on a simplification of this problem is done in [1]. Here we give one proposition showing a way (not the best way) to get coverable graphs from uncoverable graphs.

DEFINITION: Let Γ be a graph, and let G be a set of nodes on Γ. The set G is a good choice set of Γ iff G meets every maximal clique at one and only one point.

DEFINITION: Let n be a node on the graph Γ. We say that n is coverable iff there is a gcs for Γ that contains n.

PROPOSITION 10.4: Let Γ be a graph, and let Γ' be the graph obtained from Γ by removing all nodes which are not coverable. If Γ' is non-empty, then Γ' is a coverable graph.

PROOF: Let a be a node of Γ'. Then a is coverable as a node of Γ. Let G be a gcs of Γ containing a. By definition of coverable, all the nodes of G are coverable, so all the nodes of G are in Γ'. So G is a subset of Γ'. Since no new connections between nodes were formed in going from Γ to Γ', the set G can meet each maximal clique of Γ' at most once. We want to show that G meets every maximal clique of Γ' exactly once.

Let C' be a maximal clique of Γ'. Let $\mathcal{C}$ be the family of all cliques of Γ which contain C'. Since $C' \in \mathcal{C}$, the family $\mathcal{C}$ is non-empty. If $C_0 \subseteq C_1 \subseteq C_2 \subseteq \ldots$ are all in $\mathcal{C}$, then we must show that $\bigcup C_i$ is in $\mathcal{C}$. First we have that $C' \subseteq \bigcup C_i$. Next we assume that x and y are in $\bigcup C_i$. Then $x \in C_{i_0}$ and $y \in C_{i_1}$, for some i_0 and i_1. Assume that $i_1 \geq i_0$. Then $x, y \in C_{i_1}$. So $x \sim y$. Thus $\bigcup C_i$ is a clique. By Zorn's Lemma, the family $\mathcal{C}$ has a maximal element C. So we have shown that there is a maximal clique, C, of Γ such that $C' \subseteq C$.

Let $\tilde{C}$ be the set of x in C such that x is coverable (as a node of Γ). We want to show that $C' = \tilde{C}$. Since $C' \subseteq \Gamma'$, every element of C' is coverable. Since $C' \subseteq C$, every element of C' is in C. Thus $C' \subseteq \tilde{C}$. If x and y are in $\tilde{C}$, then x and y are in C, so $x \sim y$. Thus $\tilde{C}$ is a clique. So by the maximality of C' we have that $C' = \tilde{C}$.

So $C' = \{x \mid x \in C \wedge x \text{ coverable}\}$. In other words, we have $C' = C \cap \Gamma'$. Now G is a gcs for Γ containing a, so G meets C. But since G is a gcs, G is a subset of Γ'. So G meets the set $C \cap \Gamma'$. So G meets C'.

Thus G is a gcs for Γ' which contains a. Q.E.D.

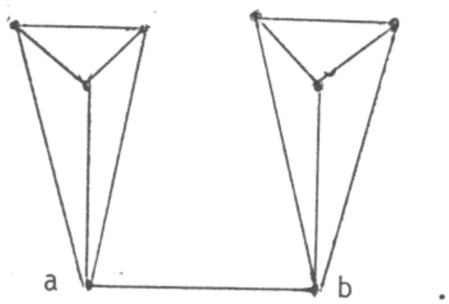
So the problem of finding coverable graphs is reduced to that of finding coverable points on graphs. Proposition 10.4 shows that this reduction is a one-step process.

The property of a weave of "having cross sets" provides a good starting point for determining which graphs represent normal weaves. We know that a weave must have cross sets in order for it to be normal. The property of having cross sets is given visual meaning in terms of graphs.

Let a and b be in Alph($\mathcal{L}$), and say there is a set L in $\mathcal{L}$ such that a, b ε L. Let L_a contain a but not b, and let L_b contain b but not a. (The sets L_a and L_b are assumed to be in $\mathcal{L}$.) Then $\langle a,b,L_a,L_b\rangle$ is a crossed quadruple, and X is a cross set for $\langle a,b,L_a,L_b\rangle$ iff $X \subseteq L_a \cup L_b - \{a,b\}$, $X \cap L_a \neq \emptyset$, and $X \cap L_b \neq \emptyset$.

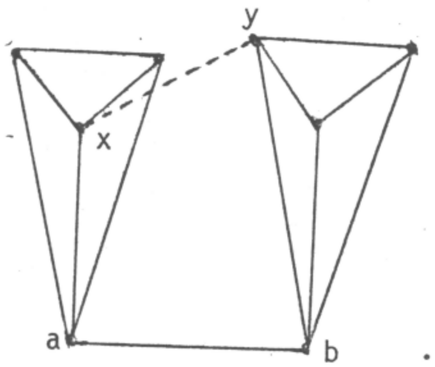
To say that a, b ε L ε $\mathcal{L}$ means that a $\approx_{\mathcal{L}}$ b, or, in pictorial terms,

To picture the sets L_a and L_b we would have something like



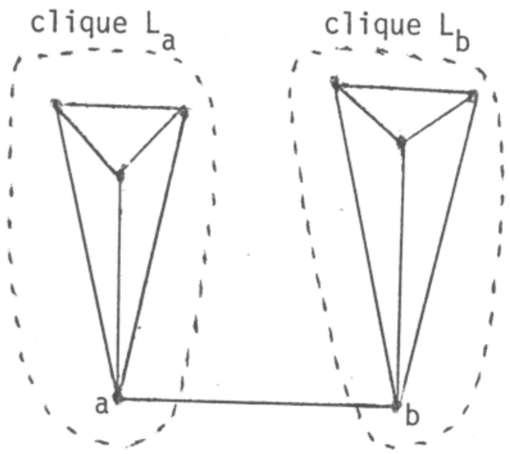
If $\mathcal{L}$ has cross sets, then for every such configuration there must be a maximal clique X meeting both L_a and L_b but not containing a or b. So there must be an

$x \in L_a$ $(x \neq a)$ and a $y \in L_b$ $(y \neq b)$ such that $x, y \in X$. This means that we must have



In other words, there must be some x in L_a $(x \neq a)$ and some y in L_b $(y \neq b)$ such that $x \approx_{\mathcal{L}} y$. This is a necessary and sufficient condition for the existance of a cross set for $\langle a,b,L_a,L_b \rangle$.

In order for a graph to represent a weave that has cross sets, there can be no configuration of the form



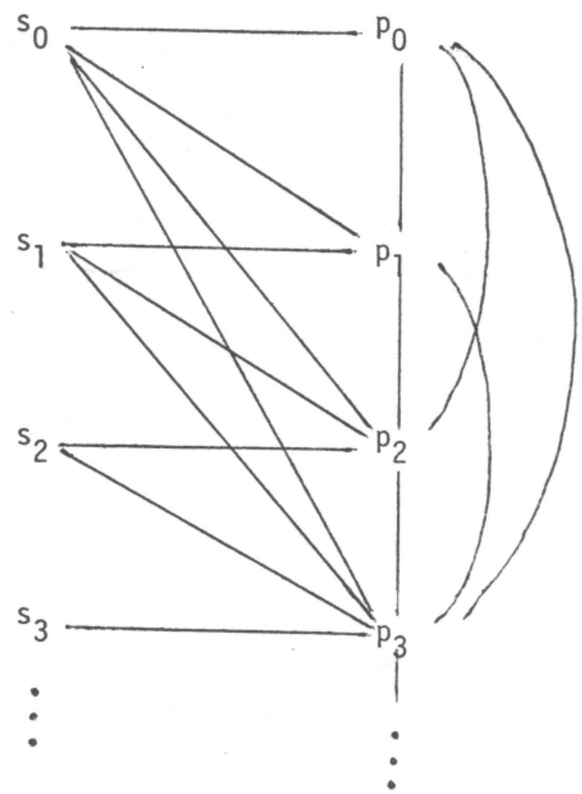
without also having a line between either

(1) some node in L_a - {a} and some node in L_b - {b},

(2) some node in L_a - {a} and the node b, or

(3) the node a and some node in L_b - {b}.

Now consider the graph



This graph represents a weave that has cross sets but is not normal. We would like a theorem which says that every such graph has a subgraph that is isomophic to this graph.

SECTION 11: DEGREES OF NORMALITY

This section is devoted to the problem: "How non-normal can a weave be?" This problem seems to have three aspects, because a weave can fail to be normal for three entirely different reasons.

(1) It can be indecomposable and not be represented by a tree. (This is something like being too disorderly.)

(2) It can be represented by a tree which is not well-founded.

(3) It can be represented by a well-founded tree that's too big to be normal. (This is the classical determinateness problem.)

It seems as if some results in area (3) would be most useful in helping solve determinateness problems. We believe that area (1) is currently the easiest in which to find results, and that results in area (1) are pre-requisites to finding results in areas (2) and (3). Although it is not stated directly, our focus in this section is on area (1). The whole subject of Normality Degrees remains an interesting and (surprisingly) complex problem.

In this section, whenever we write that $\mathcal{L}$ is $\mathcal{F}$-normal, we assume that $\mathcal{F}$ is the largest family $\mathcal{F}'$ such that $\mathcal{L}$ is $\mathcal{F}'$-normal.

DEFINITION: Let $\mathcal{L}_1$ be an $\mathcal{F}_1$-normal weave, and let $\mathcal{L}_2$ be an $\mathcal{F}_2$-normal weave. We say that $\mathcal{L}_1$ is <u>normal in</u> $\mathcal{L}_2$ iff $\mathcal{F}_1 \subseteq \mathcal{F}_2$. We say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are <u>normal equivalent</u> iff $\mathcal{L}_1$ is normal in $\mathcal{L}_2$ and $\mathcal{L}_2$ is normal in $\mathcal{L}_1$.

If we choose, we can always rename the elements of Alph($\mathcal{L}_1$) before comparing $\mathcal{L}_1$ and $\mathcal{L}_2$ for normality. In this case we'd be asking "Do $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same shape?" instead of asking "Are $\mathcal{F}_1$ and $\mathcal{F}_2$ the same?" Of course we have to be careful to note which renaming function we use, because different functions will give different answers to the question.

We give three possible definitions of "degree of normality".

DEFINITION (a): We write $\mathcal{L}_1 \in [\mathcal{L}_2]$ iff $\mathcal{L}_1$ is normal equivalent to $\mathcal{L}_2$. For any weave $\mathcal{L}_2$, the class $[\mathcal{L}_2]$ is called a <u>degree of normality</u>.

This mimics the standard definitions of Turing degree, but in our situation it seems to be more appropriate to define "degree of normality" in a slightly different (but equivalent) way. So we give another definition.

DEFINITION (*): We write $\mathcal{F}_1 \in [\mathcal{F}_2]$ iff there is a function f from Alph($\mathcal{F}_1$) into Alph($\mathcal{F}_2$) such that

(1) f is one-to-one and onto, and

(2) f induces a function f* on $\mathcal{F}_1$ which is one-to-one and onto $\mathcal{F}_2$.

(In other words, $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic as sets.)

DEFINITION (b): The class $[\mathcal{F}_2]$ is called a <u>degree of normality</u> iff there is a weave $\mathcal{L}_2$ such that $\mathcal{L}_2$ is $\mathcal{F}_2$-normal.

This is still not the definition of "degree of normality" that we will use. Our definition is slightly different. We will use definition (c) below.

DEFINITION: Let $\mathcal{L}$ be a weave. We use $\mathcal{X}(\mathcal{L})$ to denote the family of all sets X such that X is a cs of $\mathcal{L}$, but there is no G $\subseteq$ X such that G is a gcs of $\mathcal{L}$.

We choose to emphasize this family $\mathcal{X}(\mathcal{L})$ because, just as $\mathcal{G}(\mathcal{L})$ seems to be the dual of $\mathcal{L}$ in some sense ($\mathcal{L}$ and $\mathcal{G}(\mathcal{L})$ forming a pair of families that are dual to one another with regard to gcs's), so $<\mathcal{L}, \mathcal{G}(\mathcal{L}), \mathcal{X}(\mathcal{L})>$ seems, in our estimation, to form a complete triple with regard to gcs's. It is our belief that this triple $<\mathcal{L}, \mathcal{G}(\mathcal{L}), \mathcal{X}(\mathcal{L})>$ somehow encodes all data relevant to the gcs's of $\mathcal{L}$.

We let " $\mathcal{X}_1 \varepsilon [\mathcal{X}_2]$" be defined exactly as in Definition (*).

DEFINITION (c): For any weave $\mathcal{L}$, the class $[\mathcal{X}(\mathcal{L})]$ is called a <u>degree of normality</u>.

We want to characterize all families $\mathcal{X}$ that represent a degree of normality. In a similar vein, we would like to characterize all partially ordered sets $<P, \leqslant>$ that are isomorphic to some $<\mathcal{X}, \subseteq>$, where $\mathcal{X}$ represents a degree of normality. We have only a few scattered results.

PROPOSITION 11.1: If $\mathcal{X}$ represents a degree of normality, then $\mathcal{X}$ satisfies

$$\forall X, Y, Z \text{ if } X, Z \varepsilon \mathcal{X} \text{ and } X \subseteq Y \subseteq Z \text{ then } Y \varepsilon \mathcal{X}.$$

PROOF: If $\mathcal{X}$ represents a degree of normality, then $\mathcal{X} = \mathcal{X}(\mathcal{L})$ for some weave $\mathcal{L}$. So, for each X in $\mathcal{X}$, X is a cs of $\mathcal{L}$, but there is no G $\subseteq$ X such that G is a gcs of $\mathcal{L}$. Let X, Z ε $\mathcal{X}$ and X $\subseteq$ Y $\subseteq$ Z. Since Y $\supseteq$ X and X is a cs of $\mathcal{L}$, Y is a cs of $\mathcal{L}$. If there is a set G $\subseteq$ Y such that G is a gcs of $\mathcal{L}$, then G $\subseteq$ Z, contradicting the fact that Z ε $\mathcal{X}$. Thus there is no such G. So Y ε $\mathcal{X}$. Q.E.D.

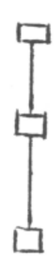
Now the task is to look for families < $\mathcal{X}$, $\subseteq$ > which satisfy the property of Proposition 11.1 but do not represent degrees of normality. So far no such example is known. For the remainder of this section we give several examples illustrating families which do represent degrees of normality.

(1) There is a family $\mathcal{X}(\mathcal{L})$ whose partial ordering < $\mathcal{X}(\mathcal{L})$, $\subseteq$ > is isomorphic to the one-point ordering. Let $\mathcal{L}$ be {{a,b}, {c,d}, {a,d}}. Then $\mathcal{G}(\mathcal{L})$ is {{a,c}, {b,d}} and $\mathcal{X}(\mathcal{L})$ is {{a,d}}.

(2) There is a family $\mathcal{X}(\mathcal{L})$ whose partial ordering < $\mathcal{X}(\mathcal{L})$, $\subseteq$ > is

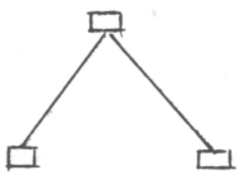
Let $\mathcal{L}$ = {{a,x}, {a,z,b}, {b,y}}. Then $\mathcal{G}(\mathcal{L})$ = {{a,y}, {x,z,y}, {x,b}}, and $\mathcal{X}(\mathcal{L})$ = {{a,b}, {a,b,z}}.

(3) The ordering



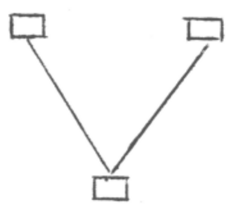
is impossible, because this would violate Proposition 11.1.

(4) There is a family $\chi(\mathcal{L})$ whose partial ordering $< \chi(\mathcal{L}), \subseteq >$ has the shape



Let $\mathcal{L} = \{\{a,b\}, \{c,d\}, \{a,d\}, \{x,y\}\}$. Then $\mathcal{G}(\mathcal{L}) = \{\{a,c,x\}, \{a,c,y\}, \{b,d,x\}, \{b,d,y\}\}$, and $\chi(\mathcal{L}) = \{\{x,a,d\}, \{y,a,d\}, \{x,y,a,d\}\}$. By taking the family $\{\{a,b\}, \{c,d\}, \{a,d\}\}$ and adding the set S to it, we can get $\chi(\mathcal{L})$ to be the same shape as $\mathcal{P}(S) - \{\emptyset\}$.

(5) There is a family $\chi(\mathcal{L})$ whose partial ordering has the shape



Let $\mathcal{L} = \{\{a,c\}, \{a,e\}, \{e,f\}, \{e,g\}\}$. Then $\mathcal{G}(\mathcal{L}) = \{\{a,f,g\}, \{c,e\}\}$, and $\chi(\mathcal{L}) = \{\{a,e\}, \{a,e,f\}, \{a,e,g\}\}$.

(6) For any set S, there is a weave $\mathcal{L}$ such that $\chi(\mathcal{L}) = \mathcal{P}(S)$. Let $\mathcal{L} = \{\{a,x\}, \{b,y\}, \{a,b\} \cup S\}$, where $a, b, x, y \notin S$. Then $\mathcal{G}(\mathcal{L})$ is $\{a,y\}$, $\{b,x\}$, and all sets of the form $\{x,y\} \cup S_0$, where $S_0 \subseteq S$. So $\chi(\mathcal{L}) = \mathcal{P}(S)$.

(7) Recall that $\mathcal{C}(\mathcal{L})$ is the family consisting of all cs's of $\mathcal{L}$. We prove that if $\mathcal{L} = \mathcal{L}_1 \dot{\vee} \mathcal{L}_2$, then

$$\chi(\mathcal{L}) = (\chi(\mathcal{L}_1) \dot{\wedge} \mathcal{C}(\mathcal{L}_2)) \vee (\mathcal{C}(\mathcal{L}_1) \dot{\wedge} \chi(\mathcal{L}_2)).$$

$\supseteq$: Let $X_1 \in \mathcal{X}(\mathcal{L}_1)$, and $C_2 \in \mathcal{C}(\mathcal{L}_2)$. Since X_1 is a cs of $\mathcal{L}_1$, and C_2 is a cs of $\mathcal{L}_2$, $X_1 \cup C_2$ is a cs of $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$. If A is a gcs of $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$, and $A \subseteq X_1 \cup C_2$, then $A \cap \text{Alph}(\mathcal{L}_1)$ is a gcs of $\mathcal{L}_1$, and $A \cap \text{Alph}(\mathcal{L}_1) \subseteq X_1$, contradicting the assumption that $X_1 \in \mathcal{X}(\mathcal{L}_1)$. Thus $X_1 \cup C_2 \in \mathcal{X}(\mathcal{L})$. Similarly, if $X_2 \in \mathcal{X}(\mathcal{L}_2)$ and $C_1 \in \mathcal{C}(\mathcal{L}_1)$ then $X_2 \cup C_1 \in \mathcal{X}(\mathcal{L})$. Thus $\mathcal{X}(\mathcal{L}) \supseteq (\mathcal{X}(\mathcal{L}_1) \dot{\wedge} \mathcal{C}(\mathcal{L}_2)) \vee (\mathcal{C}(\mathcal{L}_1) \dot{\wedge} \mathcal{X}(\mathcal{L}_2))$.

$\subseteq$: Let $X \in \mathcal{X}(\mathcal{L})$. The set X is equal to $X_1 \cup X_2$, where $X_1 \subseteq \text{Alph}(\mathcal{L}_1)$ and $X_2 \subseteq \text{Alph}(\mathcal{L}_2)$. Since X is a cs for $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$, X_1 must be a cs for $\mathcal{L}_1$, and X_2 must be a cs for $\mathcal{L}_2$. But assume that there are sets $G_1 \subseteq X_1$ and $G_2 \subseteq X_2$ such that G_1 is a gcs for $\mathcal{L}_1$ and G_2 is a gcs for $\mathcal{L}_2$. Then $G_1 \cup G_2 \subseteq X_1 \cup X_2$, and $G_1 \cup G_2$ is a gcs for $\mathcal{L}$, contradicting the assumption that $X \in \mathcal{X}(\mathcal{L})$. So either $X_1 \in \mathcal{X}(\mathcal{L}_1)$, or $X_2 \in \mathcal{X}(\mathcal{L}_2)$. Thus $X \in (\mathcal{X}(\mathcal{L}_1) \dot{\wedge} \mathcal{C}(\mathcal{L}_2)) \vee (\mathcal{C}(\mathcal{L}_1) \dot{\wedge} \mathcal{X}(\mathcal{L}_2))$.
Q.E.D.

As an application, let $\mathcal{L}_1$ be a normal weave, and $\mathcal{L}_2$ be the weave of example (1). Then $\mathcal{X}(\mathcal{L}_1 \dot{\vee} \mathcal{L}_2) = (\mathcal{C}(\mathcal{L}_1) \dot{\wedge} \mathcal{X}(\mathcal{L}_2)) \vee (\mathcal{X}(\mathcal{L}_1) \dot{\wedge} \mathcal{C}(\mathcal{L}_2))$. Since $\mathcal{L}_1$ is normal, $\mathcal{X}(\mathcal{L}_1) = \emptyset$, thus $\mathcal{X}(\mathcal{L}_1 \dot{\vee} \mathcal{L}_2) = \mathcal{C}(\mathcal{L}_1) \dot{\wedge} \mathcal{X}(\mathcal{L}_2)$, but this is isomorphic, as a partial ordering, to $\mathcal{C}(\mathcal{L}_1)$. Thus if $\langle P, \leq \rangle$ is the partial ordering of the family of cs's of a normal weave, then there is a family $\mathcal{X}(\mathcal{L})$ whose partial ordering $< \mathcal{X}(\mathcal{L}), \subseteq >$ is isomophic to $<P, \leqslant>$.

(8) For any weave $\mathcal{L}$ let $\overline{\mathcal{C}(\mathcal{L})}$ be the family $\mathcal{P}(\mathcal{L}) - \mathcal{C}(\mathcal{L})$. Then we have the formula

$$\mathcal{X}(\mathcal{L}_1 \dot{\wedge} \mathcal{L}_2) = \left[\mathcal{X}(\mathcal{L}_1) \dot{\wedge} (\mathcal{X}(\mathcal{L}_2) \vee \overline{\mathcal{C}(\mathcal{L}_2)}) \right] \vee$$

$$\left[(\mathcal{X}(\mathcal{L}_1) \vee \overline{\mathcal{C}(\mathcal{L}_1)}) \dot{\wedge} \mathcal{X}(\mathcal{L}_2) \right].$$

The proof is similar to that of example (7).

We would like to know whether there is a weave $\mathcal{L}$ such that $\mathcal{X}(\mathcal{L})$ is equal to the family of all infinite subsets of the natural numbers.

SECTION 12: THE SYSTEM LAb

In this section we present a propositional logic and give a proof system for it. Our logic will be different from the standard propositional calculus in that the subformula relation will not be well-founded. Our statements will not be defined by induction, and the proofs of some statements will be infinitely long.

We begin with a set A. If $a \varepsilon A$, then $\overline{a} \varepsilon \overline{A}$, $\overline{\overline{a}} \varepsilon \overline{\overline{A}}$, etc.

If $p \varepsilon A \cup \overline{A} \cup \overline{\overline{A}} \cup \ldots$, then p, ~p, ~~p, ... etc. are called <u>basic semi-statements</u>. The set of basic semi-statements is denoted B.

Let $\mathcal{L}$ be a normal stopped weave whose alphabet is a subset of B. Then $\mathcal{L}$ is a <u>formula</u>. If $\mathcal{L} = \{p\}$, where $p \varepsilon A \cup \overline{A} \cup \overline{\overline{A}} \cup \ldots$, then $\mathcal{L}$ is called an <u>atomic formula</u>.

By the Fundamental Theorem of Normal Weaves, every non-atomic formula $\mathcal{L}$ can be written non-trivally in one of the two forms $\mathcal{L}_1 \dot{\vee} \mathcal{L}_2$ or $\mathcal{L}_1 \dot{\wedge} \mathcal{L}_2$. In addition we define $\sim\mathcal{L}$. Let L be a set. Then $\neg L$ is defined to be $\{\sim \ell \mid \ell \ \varepsilon L\}$. For a family $\mathcal{L}$, we define $\neg\mathcal{L}$ to be $\{\neg L \mid L \varepsilon \mathcal{L}\}$. For a formula $\mathcal{L}$, we define $\sim\mathcal{L}$ to be $\mathcal{G}(\neg\mathcal{L})$.

We define the semantics of our system. Let ψ be a function from A into {T,F}. Let $\psi(a) = \psi(\overline{a}) = \psi(\overline{\overline{a}}) = \ldots$, for all a in A. By induction define

$$\psi(\sim c) = \begin{cases} T \text{ if } \psi(c) = F, \text{ and} \\ F \text{ if } \psi(c) = T. \end{cases}$$

Let $\psi \models \mathcal{L}$ iff there exists a set L in $\mathcal{L}$ such that, for all ℓ in L, $\psi(\ell) = T$.

A <u>sequent</u> is an expression of the form $\Pi \to \Delta$, where Π and Δ are finite sequences of formulas. The <u>antecedent</u> of $\Pi \to \Delta$ is Π, and the <u>succeedent</u> of $\Pi \to \Delta$ is Δ. Let $\psi \models \Pi \to \Delta$ iff there exists an $\mathcal{L}$ in Π such that $\psi \not\models \mathcal{L}$, or there exists an $\mathcal{L}$ in Δ such that $\psi \models \mathcal{L}$. A sequent $\Pi \to \Delta$ is <u>valid</u> iff $\psi \models \Pi \to \Delta$, for every ψ.

We are now ready to define a proof system. The system has "axioms" and "rules of inference", but the definition of what constitutes a proof is non-standard. This definition is equivalent to the usual definition if we restrict ourselves to the ordinary propositional calculus where the subformula relation is well-founded.

Let $\mathcal{L}$ and $\mathcal{L}'$ be formulas. If $\mathcal{L}'$ can be obtained from $\mathcal{L}$ by adding or deleting bars from some occurrances of letters that appear in $\mathcal{L}$, then $\mathcal{L}'$ is a <u>bar variant</u> of $\mathcal{L}$. (For example, $\{\{a, \bar{b}\}, \{\bar{a}, c\}\}$ is a bar variant of $\{\{a, \bar{\bar{b}}\}, \{a, c\}\}$.) If $\mathcal{L}'$ is a bar variant of $\mathcal{L}$, then $\mathcal{L} \to \mathcal{L}'$ is an axiom.

We list the rules of inference of LAb. First we have the structural rules.

(1) The weakening rules are

$$\text{left} \quad \frac{\Pi \to \Delta}{\mathcal{L}, \Pi \to \Delta}, \quad \text{and} \quad \text{right} \quad \frac{\Pi \to \Delta}{\Pi \to \Delta, \mathcal{L}}.$$

(2) The contraction rules are

$$\text{left} \quad \frac{\mathcal{L}, \mathcal{L}, \Pi \to \Delta}{\mathcal{L}, \Pi \to \Delta}, \quad \text{and} \quad \text{right} \quad \frac{\Pi \to \Delta, \mathcal{L}, \mathcal{L}}{\Pi \to \Delta, \mathcal{L}}.$$

The occurrances of $\mathcal{L}$ in the upper sequents of these inferences are called <u>predecessors</u> of the occurrance in the lower sequent.

(3) The exchange rules are

$$\text{left} \quad \frac{\Pi, \mathcal{L}_1, \mathcal{L}_2, \Lambda \to \Delta}{\Pi, \mathcal{L}_2, \mathcal{L}_1, \Lambda \to \Delta}, \text{ and}$$

$$\text{right} \quad \frac{\Pi \to \Lambda, \mathcal{L}_1, \mathcal{L}_2, \Delta}{\Pi \to \Lambda, \mathcal{L}_2, \mathcal{L}_1, \Delta}.$$

In each case the occurrance of $\mathcal{L}_1(\mathcal{L}_2)$ in the upper sequent is the predecessor of the occurrance of $\mathcal{L}_1(\mathcal{L}_2)$ in the lower sequent.

Next we have the logical rules.

(4) The negation rules are

$$\sim\text{:left} \quad \frac{\Pi \to \Delta, \mathcal{L}}{\sim\mathcal{L}, \Pi \to \Delta}, \text{ and}$$

$$\sim\text{:right} \quad \frac{\mathcal{L}, \Pi \to \Delta}{\Pi \to \Delta, \sim\mathcal{L}}.$$

In each case, $\mathcal{L}$ is the predecessor of $\sim\mathcal{L}$.

(5) The conjunction rules are

$$\wedge\text{:left} \quad \frac{\mathcal{L}_1, \Pi \to \Delta}{\mathcal{L}_1 \wedge \mathcal{L}_2, \Pi \to \Delta} \quad \text{or} \quad \frac{\mathcal{L}_2, \Pi \to \Delta}{\mathcal{L}_1 \wedge \mathcal{L}_2, \Pi \to \Delta},$$

$$\text{and} \quad \wedge\text{: right} \quad \frac{\Pi \to \Delta, \mathcal{L}_1 \qquad \Pi \to \Delta, \mathcal{L}_2}{\Pi \to \Delta, \mathcal{L}_1 \wedge \mathcal{L}_2}.$$

In $\wedge$:left either $\mathcal{L}_1$ or $\mathcal{L}_2$ is the predecessor of $\mathcal{L}_1 \wedge \mathcal{L}_2$. In $\wedge$:right both $\mathcal{L}_1$ and $\mathcal{L}_2$ are predecessors of $\mathcal{L}_1 \wedge \mathcal{L}_2$.

(6) The disjunction rules are

$$\vee\text{:left} \quad \frac{\mathcal{L}_1, \Pi \to \Delta \qquad \mathcal{L}_2, \Pi \to \Delta}{\mathcal{L}_1 \vee \mathcal{L}_2, \Pi \to \Delta}, \text{ and}$$

$$\vee\text{:right} \quad \frac{\Pi \to \Delta, \mathcal{L}_1}{\Pi \to \Delta, \mathcal{L}_1 \vee \mathcal{L}_2}, \text{ or } \frac{\Pi \to \Delta, \mathcal{L}_2}{\Pi \to \Delta, \mathcal{L}_1 \vee \mathcal{L}_2}.$$

In $\vee$:left, both $\mathcal{L}_1$ and $\mathcal{L}_2$ are predecessors of $\mathcal{L}_1 \vee \mathcal{L}_2$. In $\vee$:right, either $\mathcal{L}_1$ or $\mathcal{L}_2$ is a predecessor of $\mathcal{L}_1 \vee \mathcal{L}_2$.

For a sequent $\Pi \to \Delta$ we define a game Γ.

Stage 0: This is divided into cases.

Case (1) If $\Pi \to \Delta$ is an axiom, then the game is ended.

Case (2) There is a sequent S_0, or a pair of sequents S_1, S_2, from which $\Pi \to \Delta$ follows by a rule of inference. Then player I chooses such a sequent S_0 or such a pair S_1, S_2.

Case (3) None of the above happen. Then the game is ended.

Stage 1: If Case (2) happened in Stage 0, and player I chose S_0, then player II chooses S_0. The sequent S_0 is said to be "in play" at the end of Stage 1. If Case (2) happened in Stage 0, and player I chose S_1, S_2, then player II choses either S_1 or S_2. Whichever sequent player II choses is said to be "in play" at the end of Stage 1.

Assume that the sequent $\tilde{S}$ is in play at the end of Stage 2n - 1.

Stage 2n: This is the same as Stage 0, except we use $\tilde{S}$ instead of $\Pi \to \Delta$.

Stage 2n+1: This is the same as Stage 1, except we use the result of Stage 2n.

We now describe what happens at Stage λ, for a limit ordinal λ. Let $\tilde{S}_1, \tilde{S}_3, \tilde{S}_5, \ldots, \tilde{S}_\alpha, \ldots$ be the sequents that were in play at the end of stages 1, 3, 5, ..., α, ..., for $\alpha < \lambda$. We define the sequent $\tilde{S}_\lambda$ in the following way. Let $\mathcal{L}_1$ be a formula appearing in $\tilde{S}_1$. Let $\mathcal{L}_3$ be the predecessor of $\mathcal{L}_1$ in the sequent $\tilde{S}_3$. Let $\mathcal{L}_5$ be the predecessor of $\mathcal{L}_3$ in $\tilde{S}_5$, and so on. For each α, let $\mathcal{L}_\alpha{}'$ be the same as $\mathcal{L}_\alpha$, except with all negation symbols erased. Since $\mathcal{L}_1$ is stopped, it must be true that

$$\mathcal{L}_1{}' \upharpoonright \bigcap_{\substack{\alpha < \lambda \\ \alpha \text{ odd}}} \text{Alph}(\mathcal{L}_\alpha{}') \text{ is non-empty. Let } a \in \bigcap \text{Alph}(\mathcal{L}_\alpha{}'). \text{ Now}$$

a must appear, in some form (i.e.,with a finite number of negation signs in front of it) in $\mathcal{L}_1$. Since we can only have a <u>finite</u> number of negation signs in front of any letter, this means that the rules ~:left and ~:right could only have been applied <u>finitely many times</u> to $\mathcal{L}_1, \mathcal{L}_3, \ldots$ in the branch $\tilde{S}_1, \tilde{S}_3, \ldots$. Thus there is an ordinal $\beta < \lambda$ such that, for all γ satisfying $\beta \le \gamma < \lambda$, $\tilde{S}_\gamma$ is not obtained from $\tilde{S}_{\gamma+2}$ by applying a negation rule to $\mathcal{L}_{\gamma+2}$. (In fact, since there are only finitely many formulas in the sequent $\tilde{S}_1$, we know that the negation rules ~:left and ~:right can only be used finitely many times on the partial branch $\tilde{S}_1, \tilde{S}_3, \ldots$, and this applies to <u>any</u> partial branch.) Thus if $\mathcal{L}_\beta$ is in the antecedent of $\tilde{S}_\beta$, then $\mathcal{L}_\gamma$ is in the antecedent of $\tilde{S}_\beta$ for each γ satisfying $\beta \leq \gamma < \lambda$. (Similarly, if $\mathcal{L}_\beta$ is in the succeedent of $\tilde{S}_\beta$, then $\mathcal{L}_\gamma$ is in the succeedent of $\tilde{S}_\gamma$, for each γ satisfying $\beta \leq \gamma < \lambda$.) Furthermore we know that $\mathcal{L}_\beta \restriction \bigcap_{\substack{\beta \leq \gamma < \lambda \\ \gamma \text{ odd}}} \text{Alph}(\mathcal{L}_\gamma)$ is non-empty, since $\mathcal{L}_\beta$ is stopped. Let this be the formula $\mathcal{L}_\lambda$. Let it appear in either the antecedent or the succeedent of $\tilde{S}_\lambda$, depending on where $\mathcal{L}_\beta$ appears in $\tilde{S}_\beta$.

Stage λ (λ is a limit ordinal): Stage λ is played exactly as Stage 0, except using $\tilde{S}_\lambda$ instead of $\Pi \to \Lambda$.

In the game Γ, a play p is winning for I iff an axiom appears somewhere along p. A proof of $\Pi \to \Lambda$ is a game Γ where player I has a winning strategy.

It is not necessary to use game terminology to define a proof. Our definition of "proof" is equivalent to the following, somewhat more conventional, formulation.

Let $\mathcal{T}$ be a tree of sequents. The tree $\mathcal{T}$ is a proof of $\Pi \to \Delta$, where $\Pi \to \Delta$ is the sequent that appears on the bottom of $\mathcal{T}$, iff $\mathcal{T}$ satisfies the following conditions.

(0) Every branch has a topmost sequent.

(1) The topmost sequents of $\mathcal{T}$ are axioms.

(2) Let S occur on $\mathcal{T}$ at a node whose level is a successor ordinal. Then S is an upper sequent of an inference whose lower sequent is also on $\mathcal{T}$.

(3) Let S occur on $\mathcal{T}$ at a node whose level is a limit ordinal. Then S is obtained from its successors $\tilde{S}_1$, $\tilde{S}_3$, $\tilde{S}_5$, ... in the manner described above.

We will present a proof of soundness and completeness for the system LAb.

THEOREM 12.1: The system LAb is sound.

PROOF: Let $\Pi \to \Delta$ be a sequent which is not valid, and let $\mathcal{T}$ be a tree of sequents satisfying conditions (0), (2), and (3) given above. We will show that $\mathcal{T}$ does not satify condition (1). More specifically, we will show that there is a branch b of $\mathcal{T}$ such that no sequent on b is valid.

Since $\Pi \to \Delta$ is not valid, we know that there is a truth function ψ such that $\psi \models \bigwedge \Pi$ and $\psi \not\models \bigvee \Delta$. So for each formula $\mathcal{L}$ in Π, we can pick a set L in $\mathcal{L}$ such that $\psi(\ell) = T$, for all ℓ in L. Furthermore, for each formula $\mathcal{M}$ in Δ, we can pick a set $G \in \mathcal{G}(\mathcal{M})$ such that $\psi(g) = F$ for all g in G. These sets testify to the invalidity of $\Pi \to \Delta$. We will find consecutive predecessors $\Pi_1 \to \Delta_1$, $\Pi_2 \to \Delta_2$, ... of the sequent $\Pi \to \Delta$, and find sets which testify to their invalidity.

Case(1) Let $\Pi \to \Delta$ be obtained, on the tree, from $\Pi_1 \to \Delta_1$ by an application of the rule ~: left; i.e., we have

$$\frac{\Pi_1 \to \Delta_1', \mathcal{L}}{\sim \mathcal{L}, \Pi' \to \Delta} \,.$$

Let L_0, L_1, ..., L_p, G_0, G_1, ..., G_q testify to the invalidity of $\sim \mathcal{L}, \Pi' \to \Delta$, and let $L_0 \in \sim \mathcal{L}$. Let $L_0{}^*$ be such that $L_0 = \neg L_0{}^*$. Then L_1, ..., L_p, G_0, G_1, ..., G_q, $L_0{}^*$ testify to the invalidity of $\Pi_1 \to \Delta_1', \mathcal{L}$.

Case(2) The sequent $\Pi \to \Delta$ is obtained, on the tree, from $\Pi_1 \to \Delta_1$ by an application of ~:right. This is similar to Case(1).

Case(3) Let $\Pi \to \Delta$ be obtained, on the tree, from $\Pi_1 \to \Delta_1$ by an application of $\wedge$:left, i.e., we have

$$\frac{\mathcal{L}', \Pi_1' \to \Delta_1}{\mathcal{L}' \wedge \mathcal{L}'', \Pi' \to \Delta}.$$

Let $L_0, L_1, \ldots, L_p, G_0, \ldots, G_q$ testify to the invalidity of $\mathcal{L}' \wedge \mathcal{L}''$, $\Pi' \to \Delta$, and let $L_0 \in \mathcal{L}' \wedge \mathcal{L}''$. Then $L_0 \cap \mathrm{Alph}(\mathcal{L}')$, $L_1, \ldots, L_p, G_0, \ldots, G_q$ testify to the invalidity of $\mathcal{L}', \Pi_1' \to \Delta_1$.

Case (4) Let $\Pi \to \Delta$ be obtained, on the tree, from $\tilde{\Pi} \to \tilde{\Delta}$ and $\tilde{\tilde{\Pi}} \to \tilde{\tilde{\Delta}}$ by an application of $\wedge$: right, i.e., we have

$$\frac{\tilde{\Pi} \to \tilde{\Delta}', \mathcal{L}' \qquad \tilde{\tilde{\Pi}} \to \tilde{\tilde{\Delta}}', \mathcal{L}''}{\Pi \to \Delta', \mathcal{L}' \wedge \mathcal{L}''}.$$

Let $L_0, L_1, \ldots, L_p, G_0, \ldots, G_q$ testify to the invalidity of $\Pi \to \Delta'$, $\mathcal{L}' \wedge \mathcal{L}''$, and let $G_q \in \mathcal{G}(\mathcal{L}' \wedge \mathcal{L}'')$. Then either $G_q \in \mathcal{G}(\mathcal{L}')$ or $G_q \in \mathcal{G}(\mathcal{L}'')$. Assume without loss of generality that $G_q \in \mathcal{G}(\mathcal{L}')$. Then let $\tilde{\Pi} \to \tilde{\Delta}', \mathcal{L}'$ be $\Pi_1 \to \Delta_1$. (We have chosen a particular predecessor of $\Pi \to \Delta$.) Then $L_0, L_1, \ldots, L_p, G_0, \ldots, G_q$ testifies to the invalidity of $\Pi_1 \to \Delta_1$.

Case(5) For $\vee$: left we have something similar to Case(4). For $\vee$:right we have something similar to Case(3). All other cases are straight forward.

Now assume that we have chosen the sequents $\Pi \to \Delta$, $\Pi_1 \to \Delta_1$, $\Pi_2 \to \Delta_2, \ldots$, $\Pi_n \to \Delta_n, \ldots$ $n < \omega$ and $L_0^n, \ldots, L_p^n, G_0^n, \ldots, G_q^n$, for each n. Along this sequence find a number n_0 such that, for each k satisfying $n_0 \leq k < \omega$, $\Pi_k \to \Delta_k$ is not obtained from $\Pi_{k+1} \to \Delta_{k+1}$ by applying a negation rule to

$\Pi_{k+1} \rightarrow \Delta_{k+1}$. By choice of the sets $L_0^k, \ldots, L_p^k, G_0^k, \ldots G_q^k$, we have that

$$L_0^k \subseteq L_0^{n_0},$$

$$L_1^k \subseteq L_1^{n_0},$$

$$\vdots$$

$$G_q^k \subseteq G_q^{n_0},$$

for all k such that $n_0 \leq k < \omega$. Thus, for example $L_1^{n_0} \cap \mathrm{Alph}(\mathcal{L}_k) = L_1^k$ if $\mathcal{L}_k$ is the appropriate predecessor formula in $\Pi_k \rightarrow \Delta_k$. So $L_1^{n_0} \cap \mathrm{Alph}(\mathcal{L}_k)$ is non-empty. By applying Lemma 3.8 to this situation, we have that $L_1^{n_0} \cap \mathrm{Alph}(\mathcal{L}_\omega) \neq \emptyset$, where $\mathcal{L}_\omega$ is the appropriate predecessor formula in $\Pi_\omega \rightarrow \Delta_\omega$. Doing this for $L_2^{n_0} \cap \mathrm{Alph}(\mathcal{L}_{2_\omega}), \ldots, G_q^{n_0} \cap \mathrm{Alph}(\mathcal{M}_{q_\omega})$, we get that the collection of sets $L_1^{n_0} \cap \mathrm{Alph}(\mathcal{L}_\omega), L_2^{n_0} \cap \mathrm{Alph}(\mathcal{L}_{2_\omega}), \ldots, G_q^{n_0} \cap \mathrm{Alph}(\mathcal{M}_{q_\omega})$ testifies to the invalidity of $\Pi_\omega \rightarrow \Delta_\omega$. Thus all sequents $\Pi \rightarrow \Delta, \Pi_1 \rightarrow \Delta_1, \Pi_2 \rightarrow \Delta_2, \ldots, \Pi_\omega \rightarrow \Delta_\omega$ are invalid. We continue by induction to find a branch b such that no sequent on b is valid. This completes the soundness proof. Q.E.D.

THEOREM 12.2: The system LAb is complete.

PROOF: Let $\Pi \to \Delta$ be a sequent. We contruct a tree for $\Pi \to \Delta$, which we call a reduction tree, and use this tree to obtain a proof of $\Pi \to \Delta$, or to show that $\Pi \to \Delta$ is not valid.

We construct the reduction tree in stages.

Stage 0: We write $\Pi \to \Delta$ at the bottom of the tree.

Stage k (Assume that k is a successor ordinal. We can define a parity on k. If n is a finite number, $n \equiv m \pmod 7$, and $k - n$ is a limit ordinal (or $k - n = 0$), then k has parity m): If every topmost sequent has a formula common to its antecedent and succeedent, then we stop. If not, we divide into cases depending on the parity of k.

(0) If the parity of k is 0, let $\Phi \to \Psi$ be any topmost sequent of the tree which was defined at stage k - 1. Let $\sim \mathcal{L}_1, \ldots, \sim \mathcal{L}_p$ be all formulas in Φ whose outermost connective is $\sim$, and to which no reduction has been applied in previous stages. Then write down $\Phi \to \Psi, \mathcal{L}_1, \ldots, \mathcal{L}_p$ above $\Phi \to \Psi$.

(1) If the partiy of k is 1, let $\Phi \to \Psi$ be any topmost sequent of the tree which was defined at stage k - 1. Let $\sim \mathcal{L}_1, \ldots, \sim \mathcal{L}_p$ be all formulas in Ψ whose outermost connective is $\sim$, and to which no reduction has been applied in previous stages. Then write down $\mathcal{L}_1, \ldots, \mathcal{L}_p, \Phi \to \Psi$ above $\Phi \to \Psi$.

(2) If the parity of k is 2, let $\Phi \to \Psi$ be any topmost sequent of the tree which was defined at stage k - 1. Let $\mathcal{L}_1 \wedge \mathcal{M}_1, \ldots, \mathcal{L}_p \wedge \mathcal{M}_p$ be all formulas in Φ whose outermost connective is $\wedge$, and to which no reduction has been applied in previous stages. Then write down $\mathcal{L}_1, \mathcal{M}_1, \ldots, \mathcal{L}_p, \mathcal{M}_p, \Phi \to \Psi$ above $\Phi \to \Psi$.

(3) Let $\mathcal{L}_1 \wedge \mathcal{M}_1, \ldots, \mathcal{L}_p \wedge \mathcal{M}_p$ be all formulas in Ψ whose outermost connective is $\wedge$, and to which no reduction has been applied. Write down

all sequents of the form $\Phi \to \Psi, \mathcal{N}_1, \ldots, \mathcal{N}_p$, where $\mathcal{N}_i$ is either $\mathcal{L}_i$ or $\mathcal{M}_i$, above $\Phi \to \Psi$.

(4) Let $\mathcal{L}_1 \vee \mathcal{M}_1, \ldots, \mathcal{L}_p \vee \mathcal{M}_p$ be all formulas in Φ whose outermost connective is $\vee$, and to which no reduction has been applied. Above $\Phi \to \Psi$ write down all sequents of the form $\mathcal{N}_1, \ldots, \mathcal{N}_p, \Phi \to \Psi$, where $\mathcal{N}_i$ is either $\mathcal{L}_i$ or $\mathcal{M}_i$.

(5) Let $\mathcal{L}_1 \vee \mathcal{M}_1, \ldots, \mathcal{L}_p \vee \mathcal{M}_p$ be all formulas in Ψ whose outermost connective is $\vee$, and to which no reduction has been applied. Above $\Phi \to \Psi$ write down $\Phi \to \Psi, \mathcal{L}_1, \mathcal{M}_1, \ldots, \mathcal{L}_p, \mathcal{M}_p$.

(6) If Φ and Ψ have any formula in common, or if none of (0) through (5) apply, then write nothing above $\Phi \to \Psi$.

Stage λ (Assume that λ is a limit ordinal): Let $\Pi \to \Delta, \Pi_1 \to \Delta_1, \Pi_2 \to \Delta_2, \ldots$ be any branch on the tree of length λ. Define $\Pi_\lambda \to \Delta_\lambda$ as in the definition of the game Γ. Write $\Pi_\lambda \to \Delta_\lambda$ at the top of this branch.

If each branch of the reduction tree for $\Pi \to \Delta$ ends with a sequent whose antecedent and succeedent contain a formula in common, we can modify the tree to form a proof of $\Pi \to \Delta$. Let $\Pi \to \Delta, \Pi_1 \to \Delta_1, \ldots, \Pi_\gamma \to \Delta_\gamma$ be a branch that does not end with such a sequent. The topmost sequent $\Pi_\gamma \to \Delta_\gamma$ has only atomic formulas. Let $\psi(p) = T$, for all $\{p\}$ in Π_γ, and let $\psi(q) = F$, for all $\{q\}$ in Δ_γ. Then $\psi \not\models \Pi_\gamma \to \Delta_\gamma$. We will show that if $\psi \models \Pi \to \Delta$, then $\psi \models \Pi_i \to \Delta_i$, for every $i < \gamma$.

Let $\psi \models \Pi \to \Delta$. Then $\psi \not\models \bigwedge \Pi$ or $\psi \models \bigvee \Delta$. So either there is a formula $\mathcal{L}_0$ in Π such that $\psi \not\models \mathcal{L}_0$, or there is a formula $\mathcal{M}_0$ in Δ such that $\psi \models \mathcal{M}_0$. So there is a formula $\mathcal{L}_0$ in Π with a gcs G_0 for $\mathcal{L}_0$ such that

$\psi(g) = F$, for each g in G_0, or there is a formula $\mathcal{M}_0$ in Δ with a set M_0 in $\mathcal{M}_0$ such that $\psi(m) = T$, for each m in M_0. The sets G_0 and M_0 are said to testify to the truth of $\Pi \to \Delta$. In a manner similar to that given in the soundness theorem, we can find, for each $i < \omega$, sets G_i and M_i that testify to the truth of $\Pi_i \to \Delta_i$. Once again we find a sequent $\Pi_{n_0} \to \Delta_{n_0}$ such that no negation reductions are used on sequents $\Pi_k \to \Delta_k$, for $k \geq n_0$. At this point we have G_{n_0} and M_{n_0} testifying to the truth of $\Pi_{n_0} \to \Delta_{n_0}$. (We assume, for simplicity, that $G_{n_0} \in \mathcal{G}(\mathcal{L}_{n_0})$ for some $\mathcal{L}_{n_0}$ in Π_{n_0}, and that $M_{n_0} \in \mathcal{M}_{n_0}$, for some $\mathcal{M}_{n_0}$ in Δ_{n_0}.) We can also show that, for each $k \geq n_0$, $G_{n_0} \cap \mathrm{Alph}(\mathcal{L}_k) = G_k \neq \emptyset$ and $M_{n_0} \cap \mathrm{Alph}(\mathcal{M}_k) = M_k \neq \emptyset$, for the appropriate predecessor formulas $\mathcal{L}_k$ and $\mathcal{M}_k$. Thus, by Lemma 3.8, there is a formula $\mathcal{L}_\omega$ in Π_ω such that $G_{n_0} \cap \mathrm{Alph}(\mathcal{L}_\omega) = G_\omega \neq \emptyset$, and a formula $\mathcal{M}_\omega$ in Δ_ω such that $M_{n_0} \cap \mathrm{Alph}(\mathcal{M}_\omega) = M_\omega \neq \emptyset$. Thus G_ω and M_ω testify to the truth of $\Pi_\omega \to \Delta_\omega$. So $\psi \models \Pi_\omega \to \Delta_\omega$. Continuing by induction, we find that $\psi \models \Pi_\gamma \to \Delta_\gamma$, the topmost sequent. But this is a contradiction. Thus $\psi \not\models \Pi \to \Delta$, which is what we wanted to prove. Q.E.D.

SECTION 13: THE SYSTEM LJ$\mathcal{G}$

In this section we introduce a logic LJ$\mathcal{G}$ which has many of the properties of an intuitionistic system. For simplicity we have made LJ$\mathcal{G}$ a propositional logic. There is no difficulty in expanding it into a predicate logic. In much of this section we mimic constructions done in Section 12.

We begin with a set A. If $a \in A$, then $\overline{a} \in \overline{A}$, $\overline{\overline{a}} \in \overline{\overline{A}}$, etc.

If $p \in A \cup \overline{A} \cup \overline{\overline{A}} \cup \ldots$, then p and ~p are called basic semi-statements. We identify ~~p with p. (This is done here as an experiment and is not an essential or necessary part of the system.) The set of basic semi-statements is denoted B.

Let $\mathcal{L}$ be a weave whose alphabet is a finite subset of B. (Notice that $\mathcal{L}$ need not be normal.) Then $\mathcal{L}$ is a formula. If $\mathcal{L} = \{p\}$, where $p \in A \cup \overline{A} \cup \overline{\overline{A}} \cup \ldots$, then $\mathcal{L}$ is called an atomic formula.

We define connectives in LJ$\mathcal{G}$ in the following way. Let L be a set. Then $\neg L = \{\sim \ell \mid \ell \in L\}$, and $\ulcorner L = \{\{\sim \ell\} \mid \ell \in L\}$. Notice that $\ulcorner L$ is a formula rather than a set. Let $\mathcal{L}$ be a formula. Then $\neg\mathcal{L} = \{\neg L \mid L \in \mathcal{L}\}$. We define $\sim\mathcal{L}$ to be $\mathcal{G}(\neg\mathcal{L})$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be formulas. Obtain $\overline{\mathcal{L}_2}$ from $\mathcal{L}_2$ by adding, to the letters of $\mathcal{L}_2$, enough bars to insure that $\mathrm{Alph}(\mathcal{L}_1) \cap \mathrm{Alph}(\mathcal{L}_2) = \emptyset$. Define $\mathcal{L}_1 \vee \mathcal{L}_2$ by letting it equal $\mathcal{L}_1 \dot{\vee} \overline{\mathcal{L}_2}$. Define $\mathcal{L}_1 \wedge \mathcal{L}_2$ by letting it equal $\mathcal{L}_1 \dot{\wedge} \overline{\mathcal{L}_2}$. If L_1 and L_2 are sets, obtain $\overline{L}_2$ by adding enough bars to the letters of L_2 to insure that $L_1 \cap L_2 = \emptyset$. Let $L_1 \supset \overline{L}_2$ be defined as $\ulcorner L_1 \vee \{\overline{L}_2\}$. Let

$\mathcal{L}_1 \supset \mathcal{L}_2$ be defined as $\bigwedge_{L_1 \varepsilon \mathcal{L}_1} \bigvee_{L_2 \varepsilon \mathcal{L}_2} L_1 \supset \bar{L}_2$.

The semantics of our system are defined as in Section 12.

If we like we can add $\forall$ and $\exists$ to our system in the following way. We replace symbols a of our alphabet by relation symbols. If $\mathcal{L}(x)$ is a formula with x as a "free variable", we define $\forall x \mathcal{L}(x)$ to be the formula $\bigwedge_c \mathcal{L}(c)$, where c ranges over all available constants. We define $\exists x \mathcal{L}(x)$ to be $\bigvee_c \mathcal{L}(c)$, where c ranges over all available constants.

We would like to make some observations about what we have done so far. First of all, notice that $\mathcal{L} \to {\sim\sim}\mathcal{L}$ is valid for every formula $\mathcal{L}$. This is because ${\sim\sim}\mathcal{L}$ is equal to $\mathcal{G}(\neg\mathcal{G}(\neg\mathcal{L}))$, which, in turn, is equal to $\mathcal{G}\mathcal{G}(\mathcal{L})$, and $\mathcal{G}\mathcal{G}(\mathcal{L}) \supseteq \mathcal{L}$, for every formula $\mathcal{L}$. It is not the case, however, that ${\sim\sim}\mathcal{L} \to \mathcal{L}$ is valid for every $\mathcal{L}$. For instance let $\mathcal{L}$ be {{a,b}, {c,d}, {a,c}}. Then $\mathcal{G}\mathcal{G}(\mathcal{L})$ is {{a,b}, {c,d}, {a,c}, {b,d}}. We can let $\psi(b) = \psi(d) = T$ and $\psi(a) = \psi(c) = F$. Then $\psi \models {\sim\sim}\mathcal{L}$, but $\psi \not\models \mathcal{L}$, so $\psi \not\models {\sim\sim}\mathcal{L} \to \mathcal{L}$. It is in this sense that LJ$\mathcal{G}$ resembles an intuitionistic system. If $\mathcal{L}$ is normal, then $\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{L}$. So, for every normal weave $\mathcal{L}$, we know that ${\sim\sim}\mathcal{L} \to \mathcal{L}$ is valid. However, $\mathcal{L}$ need not be normal in order for $\mathcal{G}\mathcal{G}(\mathcal{L})$ to equal $\mathcal{L}$. Let $\mathcal{L}$ be the weave {{a,x}, {a,d,b}, {b,y}}. Then $\mathcal{G}(\mathcal{L})$ = {{a,y}, {x,d,y}, {b,x}}, and $\mathcal{G}\mathcal{G}(\mathcal{L}) = \mathcal{L}$. However $\mathcal{L}$ is not normal, because {a,b} is a cs of $\mathcal{L}$, and {a,b} is not a superset of any set in $\mathcal{G}(\mathcal{L})$. We can show, using cross sets, that there is no normal weave $\mathcal{L}'$ such that $\mathcal{L} \subseteq \mathcal{L}'$. If $\mathcal{L} \subseteq \mathcal{L}'$, then <a,b, {a,x}, {b,y}> is a crossed quadruple of $\mathcal{L}'$. Since we expect $\mathcal{L}'$ to be normal, it must have cross sets. So the set {x,y} must be in $\mathcal{L}'$. But then {a,x}, {b,y}, and {x,y} are sets in $\mathcal{L}'$ so $\langle x,y, \{a,x\}, \{b,y\}\rangle$ is a crossed quadruple of $\mathcal{L}'$. Thus the set $\{a,b\}$ must be in $\mathcal{L}'$. But then we have $\{a,b\}$ and $\{a,b,d\}$ both in $\mathcal{L}'$, and this violates the Corollary to Lemma 1.1.

We need to define a property for formulas that is analogous to normality for weaves.

DEFINITION: A set $S \subseteq B$ is <u>compatible</u> iff there is no letter a such that $\overset{\overline{\vdots}}{\bar{a}}$ (n bars) and $\sim\overset{\overline{\vdots}}{\bar{a}}$ (m bars) are both in S.

Observe that if ψ is a truth function, then $\{b \in B \mid \psi(b) = F\}$ and $\{b \in B \mid \psi(b) = T\}$ are compatible sets. Likewise, by setting $\psi(s) = T$, for each s in S (or by setting $\psi(s) = F$ for each s in S), we get a truth-function from a compatible set S.

DEFINITION: A formula $\mathcal{L}$ is <u>regular</u> iff, for each cs C of $\mathcal{L}$, if C is compatible, then there is a set $G \subseteq C$ such that $G \in \mathcal{G}(\mathcal{L})$.

We want to show that the sequent $\rightarrow \mathcal{L}, \sim\mathcal{L}$ is valid iff $\mathcal{L}$ is regular. Let $\mathcal{L}$ be regular, and let ψ be a truth-function. Assume that $\psi \nvDash \mathcal{L}$. Then $\{ \ell \in \mathrm{Alph}(\mathcal{L}) \mid \psi(\ell) = F\}$ is a compatible cs of $\mathcal{L}$. By the regularity of $\mathcal{L}$, this set has a subset G such that $G \in \mathcal{G}(\mathcal{L})$. So $\psi(g) = F$, for each g in G. So $\psi(h) = T$, for each h in $\neg G$. But $\neg G \in \sim\mathcal{L}$. Therefore $\psi \vDash \sim\mathcal{L}$.

Now we assume that $\mathcal{L}$ is not regular. Let C be a compatible cs of $\mathcal{L}$ such that there is no set G in $\mathcal{G}(\mathcal{L})$ satisfying $G \subseteq C$. Define a truth function ψ by letting $\psi(c) = F$, for each c in C, and $\psi(d) = T$ otherwise. Then $\psi \nvDash \mathcal{L}$, and $\psi \nvDash \sim\mathcal{L}$. So $\rightarrow \mathcal{L}, \sim\mathcal{L}$ is not valid.

If $\mathcal{L}$ is a tautology, this means that there is no truth-function ψ such that $\psi \not\models \mathcal{L}$, so there is no compatible cs of $\mathcal{L}$. So $\mathcal{L}$ is regular.

We show that the possible rule of inference

$$\frac{\Pi \to \Delta, \mathcal{L}}{\sim \mathcal{L}, \Pi \to \Delta}$$

is sound. If $\psi \models \Pi \to \Delta, \mathcal{L}$, then either $\psi \not\models \bigwedge \Pi$ or $\psi \models \bigvee \Delta \vee \mathcal{L}$. If $\psi \not\models \bigwedge \Pi$ or $\psi \models \bigvee \Delta$, then we are done. If $\psi \models \mathcal{L}$, then there is a set L in $\mathcal{L}$ such that $\psi(\ell) = T$, for each $\ell \varepsilon L$. For any set G in $\mathcal{G}(\mathcal{L})$, $G \cap L \neq \emptyset$. Thus, for any $G \varepsilon \mathcal{G}(\mathcal{L})$, there is a letter ℓ such that $\psi(\ell) = T$. Thus, for any set H in $\mathcal{G}(\neg\mathcal{L})$, which is equal to $\sim\mathcal{L}$, there is a letter $\sim\ell$ such that $\psi(\sim \ell) = F$. So $\psi \not\models \sim\mathcal{L}$. Thus $\psi \models \sim \mathcal{L}, \Pi \to \Delta$.

We show that the possible rule of inference

$$\frac{\mathcal{L}, \Pi \to \Delta}{\Pi \to \Delta, \sim\mathcal{L}}$$

does not always hold.

DEFINITION: A formula $\mathcal{L}$ is $\mathcal{F}$-<u>regular</u> iff for each cs C of $\mathcal{L}$, if C is compatible, and $C \varepsilon \mathcal{F}$, then there is a set $G \subseteq C$ such that $G \varepsilon \mathcal{G}(\mathcal{L})$.

We have been given a sequent $\Pi \to \Delta$ and a formula $\mathcal{L}$. Let $\mathcal{F} = \{\{\ell \varepsilon \text{Alph}(\mathcal{L}) \mid \psi(\ell) = F\} \mid \psi \not\models \Pi \to \Delta\}$. We claim that the above rule of inference holds iff $\mathcal{L}$ is $\mathcal{F}$-regular. The rule holds if the following is true: for each ψ, if $\psi \models \bigwedge \Pi$ and $\psi \not\models \bigvee \Delta$, then either $\psi \models \sim\mathcal{L}$ or $\psi \models \mathcal{L}$. This, in turn, is true iff, for each ψ, if $\psi \not\models \Pi \to \Delta$, then either

$\psi \models \mathcal{L}$ or $\psi \models \sim\mathcal{L}$. Let $\mathcal{L}$ be $\mathcal{F}$-regular, and assume that $\psi \not\models \Pi \to \Delta$ and that $\psi \not\models \mathcal{L}$. Then $\{\ell \in \mathrm{Alph}(\mathcal{L}) \mid \psi(\ell) = F\}$ is a compatible cs of $\mathcal{L}$ and is also a member of $\mathcal{F}$. So we have a subset G of this set, and $\psi(g) = F$ for each g in G. So $\psi(h) = T$, for each h in $\neg G$. Since $\neg G$ is in $\sim\mathcal{L}$, $\psi \models \sim\mathcal{L}$.

Now assume that $\mathcal{L}$ is not $\mathcal{F}$-regular. Let C be a compatible cs of $\mathcal{L}$, such that $C \notin \mathcal{F}$, and there is no set G in $\mathcal{G}(\mathcal{L})$ satisfying $G \subseteq C$. Define ψ by letting $\psi(c) = F$, for each c in C, and $\psi(d) = T$ otherwise. Since C is not in $\mathcal{F}$, $\psi \not\models \Pi \to \Delta$. Also $\psi \not\models \mathcal{L}$, and $\psi \not\models \sim\mathcal{L}$. Thus the possible rule of inference

$$\frac{\mathcal{L}, \Pi \to \Delta}{\Pi \to \Delta, \sim\mathcal{L}}$$

does not hold for this formula $\mathcal{L}$.

We define a proof system for LJ$\mathcal{G}$.

If $\mathcal{L}$ is a formula, then $\mathcal{L} \to \mathcal{L}$ is an axiom.

The structural rules of inference are those of LAb.

The logical rules are as follows.

(1) The negation rules are

$$\sim\text{:left} \qquad \frac{\Pi \to \Delta, \mathcal{L}}{\sim\mathcal{L}, \Pi \to \Delta}, \qquad \text{and}$$

$$\sim\text{:right} \qquad \frac{\mathcal{L}, \Pi \to \Delta}{\Pi \to \Delta, \sim\mathcal{L}}.$$

We insist, in ~:right, that $\mathcal{L}$ be $\mathcal{F}$-regular, where $\mathcal{F} = \{\{\ell \in \mathrm{Alph}(\mathcal{L}) \mid \psi(\ell) = F\} \mid \psi \not\models \Pi \to \Delta\}$.

(2) There is a double negation rule. It is

$$\frac{\Pi \to \Delta, \mathcal{L}}{\Pi \to \Delta, \sim\sim\mathcal{L}} .$$

If we choose not to identify ~~p with p, for all $p \in A \cup \bar{A} \cup \bar{\bar{A}} \cup \ldots$, as we did in the beginning of this section, we must also add the rule

$$\frac{\mathcal{L}, \Pi \to \Delta}{\sim\sim \mathcal{L}, \Pi \to \Delta} ,$$

for all formulas $\mathcal{L}$ satisfying $\mathcal{L} = \mathcal{G}\mathcal{G}(\mathcal{L})$.

(3) The conjunction and disjunction rules are the same as in LAb.

(4) The implication rules are

$\supset$:left $$\frac{\Pi \to \Delta, \mathcal{L}_1 \quad \mathcal{L}_2, \Phi \to \Psi}{\mathcal{L}_1 \supset \mathcal{L}_2, \Pi, \Phi \to \Delta, \Psi} ,$$ and

$\supset$:right $$\frac{\mathcal{L}_1, \Pi \to \Delta, \mathcal{L}_2}{\Pi \to \Delta, \mathcal{L}_1 \supset \mathcal{L}_2} .$$

(5) There are two substitution rules. They are

$$\frac{\mathcal{L}, \Pi \to \Delta}{\mathcal{L}', \Pi \to \Delta} , \text{ and } \frac{\Pi \to \Delta, \mathcal{L}}{\Pi \to \Delta, \mathcal{L}'} ,$$

where $\mathcal{L}'$ is obtained from $\mathcal{L}$ by adding or deleting some bars from occurrances of letters that appear in $\mathcal{L}$.

If we want LJ$\mathcal{G}$ to be a predicate calculus, we add the standard Gentzen-type rules for $\forall$ and $\exists$. (See [7].)

THEOREM 13.1: The system $LJ\mathcal{G}$ is sound.

This is proved by a routine induction.

THEOREM 13.2: The system $LJ\mathcal{G}$ is complete.

PROOF: Let $\Pi \to \Delta$ be a sequent. We construct a tree for $\Pi \to \Delta$ which we call a reduction tree, and use this tree to obtain a proof of $\Pi \to \Delta$, or to show that $\Pi \to \Delta$ is not valid.

We construct the reduction tree in stages.

Stage 0: We write $\Pi \to \Delta$ at the bottom of the tree.

Stage k (If $k \equiv m \pmod{11}$, then m is the parity of k.): We stop if every topmost formula has either

(i) a formula common to its antecedent and succeedent,

(ii) a formula and its negation in the antecedent,

(iii) a formula $\mathcal{L}$ in the antecedent and $\mathcal{G}\mathcal{G}(\mathcal{L})$ in the succeedent, or

(iv) a regular formula and its negation in the succeedent.

If not, we divide into cases depending on the parity of k.

(0) This is the same as the case of "parity of k equals 1" in the proof of completeness for LAb.

(1) This is the same as the case of "parity of k equals 1" in the proof of completeness for LAb, except we insist that $\sim \mathcal{L}_1, \ldots, \sim \mathcal{L}_p$ be $\mathcal{F}_i$-regular, where $\mathcal{F}_i = \{\{ \ell \in \mathrm{Alph}(\sim \mathcal{L}_i) \mid \psi(\ell) = F\} \mid \psi \not\models \Phi \to \Psi\}$, for i from 1 to p.

(2) This is the same as "parity of k equals 2" in the completeness proof of LAb.

(3), (4), and (5) These are all the same as in the completeness proof of LAb.

(6) If the parity of k is 6, let $\Phi \to \Psi$ be any topmost sequent of the tree which was defined at stage k - 1. Let $\mathcal{L}_1 \supset \mathcal{M}_1, \ldots, \mathcal{L}_p \supset \mathcal{M}_p$ be all formulas of Φ whose outermost connective is $\supset$, and to which no reduction has been applied in previous stages. Then above $\Phi \to \Psi$ write down $\Phi \to \Psi, \mathcal{L}_1$, $\Phi \to \Psi, \mathcal{L}_2, \ldots, \Phi \to \Psi, \mathcal{L}_p$, and $\mathcal{M}_1, \ldots, \mathcal{M}_p, \Phi \to \Psi$.

(7) If the parity of k is 7, let $\Phi \to \Psi$ be any topmost sequent of the tree which was defined at stage k - 1. Let $\mathcal{L}_1 \supset \mathcal{M}_1, \ldots, \mathcal{L}_p \supset \mathcal{M}_p$ be all formulas of Ψ whose outermost connective is $\supset$, and to which no reduction has been applied in previous stages. Then write $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_p, \Phi \to \Psi$, $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_p$ above $\Phi \to \Psi$.

(8) Let $\mathcal{L}_1, \ldots, \mathcal{L}_p$ be all formulas in Φ that are non-atomic and in-decomposable. Obtain $\mathcal{L}_1', \ldots, \mathcal{L}_p'$ from $\mathcal{L}_1, \ldots, \mathcal{L}_p$ respectively, by adding bars to some occurrences of letters so that $\mathcal{L}_1', \ldots, \mathcal{L}_p'$ are all decomposable. Then, above $\Phi \to \Psi$, write down $\mathcal{L}_1', \ldots, \mathcal{L}_p', \Phi^* \to \Psi$, where Φ^* is obtained from Φ by removing $\mathcal{L}_1, \ldots, \mathcal{L}_p$.

(9) Let $\mathcal{L}_1, \ldots, \mathcal{L}_p$ be all formulas in Ψ that are non-atomic and in-decomposable. Obtain $\mathcal{L}_1', \ldots, \mathcal{L}_p'$ from $\mathcal{L}_1, \ldots, \mathcal{L}_p$ respectively as in (8). Then, above $\Phi \to \Psi$, write down $\Phi \to \Psi^*, \mathcal{L}_1', \ldots, \mathcal{L}_p'$, where Ψ^* is obtained from Ψ by removing $\mathcal{L}_1, \ldots, \mathcal{L}_p$.

(10) If $\Phi \to \Psi$ satisfies any of the conditions (i), (ii), (iii), or (iv) given at the beginning of this proof, or if none of (0) through (9) apply to $\Phi \to \Psi$, then write nothing above $\Phi \to \Psi$.

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If each branch of the reduction tree for $\Pi \rightarrow \Delta$ ends with a sequent which satisfies one of the conditions (i), (ii), (iii), or (iv) given in the beginning of this proof, we can modify the tree to form a proof of $\Pi \rightarrow \Delta$. Let $\Pi \rightarrow \Delta$, $\Pi_1 \rightarrow \Delta_1, \ldots, \Pi_t \rightarrow \Delta_t$ be a branch that does not end with such a sequent. The topmost sequent $\Pi_t \rightarrow \Delta_t$ has only atomic formulas. Let $\psi(p) = T$ for all {p} in Π_t, and let $\psi(q) = F$ for all {q} in Δ_t. Then $\psi \not\models \Pi_t \rightarrow \Delta_t$. As in the completeness proof for LAb we show that if $\psi \models \Pi \rightarrow \Delta$, then $\psi \models \Pi_i \rightarrow \Delta_i$ for every i from 1 to t. Thus $\psi \not\models \Pi \rightarrow \Delta$, which is what we wanted to prove. Q.E.D.

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