THE GIANT SPIRAL

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ABSTRACT. Cocycles of Z^m actions on compact metric spaces provide a means for constructing R^m actions or flows, called suspension flows. It is known that all R^m flows with a free dense orbit have an almost one-to-one extension which is a suspension flow. When $m = 1$, examples of cocycles are easy to construct; there is a one-to-one correspondence between cocycles and real valued continuous functions. However, when $m > 1$ the construction of examples of cocycles becomes more problematic. The only existing class of examples, the close to linear cocycles, have strong linearity properties and are well understood. In fact, when the Z^m action is uniquely ergodic, all cocycles are close to linear. We will show that in general this need not be the case. We present a method, suggested to us by Hillel Furstenberg, for constructing examples of cocycles when $m > 1$ and use this method to construct a non close to linear cocycle on a minimal Z^2 action.

1. INTRODUCTION

Let X be a compact metric space and let Z^m act as a group of commuting homeomorphisms on X. That is, we have a flow (X, Z^m). For $a \in Z^m$, we denote the action of a on $x \in X$ by ax. A cocycle for the flow (X, Z^m) is a continuous map $h : X \times Z^m \to R^m$ satisfying the cocycle equation: for any $a, b \in Z^m$ and $x \in X$, $h(x, a + b) = h(x, a) + h(ax, b)$.

A cocycle $h : X \times Z^m \to R^m$ can be used to construct the suspension (X_h, R^m) of the flow (X, Z^m): we have a Z^m action on $X \times R^m$ given by

$$T_a(x, w) = (ax, w - h(x, a))$$

for $a \in Z^m$, $x \in X$, $w \in R^m$. Because h is a cocycle, it is easily verified that $T_a \circ T_b = T_{a+b}$ and hence $a \to T_a$ defines a Z^m action on $X \times R^m$. We also have a natural R^m action on $X \times R^m$ via $((x, v), w) \to (x, v + w)$ for $x \in X$, $v, w \in R^m$. These two actions commute and so the R^m flow on $X \times R^m$ gives an R^m action on X_h, where X_h is the quotient space of $X \times R^m$ modulo the Z^m action on it. When $m = 1$, the suspension flow is the usual flow under a function.

Suspensions are of interest for several reasons. First, we look to R^m suspensions as a way of generating examples of R^m flows with interesting dynamical behavior. Even more fundamentally, as the following theorem shows, suspensions play an important role in modeling general R^m flows [R 88].

Theorem 1.1. *An R^m flow (Y, R^m) on a compact metric space Y with a free dense orbit has an almost one-to-one extension which is a suspension.*

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The relationship between the properties of h and the topological properties of the resulting suspension (X_h, R^m) have been studied by Furstenberg, Keynes, Markley and Sears; readers interested in more details should consult [FKMS 93]. The cocycles of most interest are the *covering* and *embedding* cocycles. For these cocycles the resulting suspension will be a compact metric space. If h is embedding, then in addition, X can be embedded in X_h as a global section of the R^m flow. The salient feature of covering and embedding cocycles is that they have "sufficient growth":

Theorem 1.2 (FKMS 93)**.** *Let $h : X \times Z^m \to R^m$ be a cocycle. Then h is embedding (covering) if and only if there exists a constant $B > 0$ such that*

$$B|a| \leq |h(x,a)|$$

for all $a \in Z^m$ (for all $a \in Z^m$ with $|a|$ sufficiently large).

(We also note, as a consequence of the cocycle equation, given a cocycle $h : X \times Z^m \to R^m$, there exists a constant $B' > 0$ such that $|h(x,a)| \leq B'|a|$ for all $a \in Z^m$.)

The first example of cocycles are the *coboundaries*; for each continuous function $f : X \to R^m$, we obtain a cocycle via $h(x,a) = f(ax) - f(x)$. If two cocycles h and h' in $\mathcal{C}$ differ by a coboundary, they are said to be *cohomologous*. This notion is important because cohomologous cocycles result in isomorphic suspensions [FKMS 93]. We will denote the collection of coboundaries by $\mathcal{B}$.

Another collection of accessible examples is $\mathcal{L} = \{M : R^m \to R^m | M \text{ is linear } \}$. For each $M \in \mathcal{L}$ we obtain a cocycle in $\mathcal{C}$ by $h(x,a) = M(a)$. Such a cocycle is called a *constant cocycle*. The covering and embedding constant cocycles are those for which M is nonsingular [FKMS 93].

A final subset of $\mathcal{C}$ which will be of interest is the *close to linear cocycles*, denoted by $\mathcal{D}$. The close to linear cocycles are those in the closed subspace $\mathcal{L} + \overline{\mathcal{B}}$. In fact, it can be shown that $\mathcal{D}$ is the direct sum of $\mathcal{L}$ and $\overline{\mathcal{B}}$ [KMS 94]. A constant cocycle can be badly perturbed by a coboundary, let alone an arbitrary element of $\overline{\mathcal{B}}$. However, many of the consequences of linearity persist in the close to linear cocycles. That is evidenced in the following theorem from [KMS 94]:

Theorem 1.3. *Let $h \in \mathcal{D}$. Then there exists $M \in \mathcal{L}$ such that*

$$\frac{|h(x,a) - M(a)|}{|a|} \to 0$$

uniformly in x as $|a| \to \infty$.

This strong linear quality renders the close to linear cocycles a manageable class to study. A covering close to linear cocycle $h : X \times Z^m \to R^m$ can be "unfolded". The suspension it generates is a time change of the suspension generated by the identity map $I : R^m \to R^m$; that is, there is an orbit preserving homeomorphism between X_h and X_I. (X_I is known as the *constant one suspension.*)

We can view the real valued cocycle, $h_i^a : X \times Z \to R$, obtained by restricting a component cocycle of $h : X \times Z^m \to R^m$ to a direction $a \in Z^m$ as a cocycle of the integer flow (X, T_a) where $T_a : X \to X$ is the homeomorphism given by $a \in Z^m$. It is natural to ask whether we may deduce something about the growth of the real valued cocycles h_i^a from the growth of the R^m valued cocycle h. If h is a covering, close to linear cocycle, then given any bounded set $B \subset Z^m$, h is cohomologous to a cocycle h' whose images $\{h'(x, b) : b \in B\}$ are arbitrarily close to the images of M on B where M is a nonsingular linear map. Thus, because M is nonsingular, when restricting the component cocycles of h to an arbitrary direction, at least one will be covering. As our example will show, this need not be true in general, demonstrating that the "sufficient growth" required to give a covering cocycle can be the result of the combined contributions of all the component cocycles, with no single component cocycle having the growth itself.

When $m = 1$, there exists non close to linear cocycles if and only if the flow (X, Z) is not uniquely ergodic [KMS 94]. When $m > 1$, it is only known that unique ergodicity implies all cocycles are close to linear. The cocycle equation is more restrictive at higher dimensions making cocycle constructions more difficult; thus, proving the existence of a covering cocycle on a minimal flow which is not close to linear is a nontrivial problem. The goal of this work is to construct such a cocycle. In Section 2 we will describe a general technique, communicated to us by H. Furstenberg, for constructing examples of cocycles on Z^m flows. This section will end with an example of a cocycle, the giant spiral, which is embedding but not close to linear. However, the underlying Z^2 flow associated with this example consists of a single dense orbit and a circle of fixed points; as a result, the suspension built using the giant spiral is not minimal. We then ask if it is possible to have a non close to linear embedding cocycle when the underlying Z^m flow, and hence the resulting R^m suspension, is minimal. Section 3 contains the modifications of the giant spiral which are necessary to construct such an example. Section 4 contains a description of other properties of this example and what these properties tell us about cocycles and suspensions in general.

2. A Method for Cocycle Constructions

When attempting to construct an example of a cocycle, there are several issues to consider; we need to construct the underlying Z^m flow, (X, Z^m) as well as the continuous function $h : X \times Z^m \to R^m$ satisfying the cocycle equation. In addition, we may want the space (X, Z^m) to be minimal. When $m = 1$, there is a one-to-one correspondence between continuous real valued functions on X and real valued cocycles; by constructing a continuous real valued function on X, we obtained a cocycle. When $m \neq 1$, without this correspondence, given a flow (X, Z^m), construction of a continuous map $h : X \times Z^m \to R^m$ which satisfies the cocycle equation becomes more difficult.

In order to construct an R^m-valued cocycle on a Z^m-flow, we will begin with an element of $(R^m)^{Z^m}$. Let $(R^m)^{Z^m}$ have the topology of pointwise convergence and set $\mathcal{S} = \{f : Z^m \to R^m \mid f(0) = 0\} \subset (R^m)^{Z^m}$. The subset $\mathcal{S}$ is closed in $(R^m)^{Z^m}$ with respect to the topology of pointwise convergence. The mapping from $\mathcal{S} \times Z^m$ to $\mathcal{S}$ given by $af(b) = f(a+b) - f(a)$ for $f \in \mathcal{S}$ and $a, b \in Z^m$ defines a Z^m action on $\mathcal{S}$. We have a natural cocycle

$$h : \mathcal{S} \times Z^m \to R^m \quad \text{via} \quad h(g, a) = g(a)$$

for $g \in \mathcal{S}$, $a \in Z^m$. It is not difficult to verify that h will be a cocycle since

$$h(g, a+b) = g(a) + g(a+b) - g(a) = g(a) + ag(b) = h(g, a) + h(ag, b)$$

for $g \in \mathcal{S}$ and $a, b \in Z^m$; we will call this cocycle the *evaluation cocycle*. For any $f \in \mathcal{S}$ we have a flow $(\overline{\mathcal{O}(f)}, Z^m)$ with a dense orbit and, by restricting h to $\overline{\mathcal{O}(f)}$, a cocycle on it. This technique for obtaining an example of a cocycle on a flow with a dense orbit is due to Hillel Furstenberg (personal communication). A thorough discussion of the details that follow can be found in [M 94, Chapter 3].

Of course, we do not expect this construction to lead to good examples for an arbitrarily chosen $f \in \mathcal{S}$. We would like $\overline{\mathcal{O}(f)}$ to be a compact metric space and $(\overline{\mathcal{O}(f)}, Z^m)$ to be minimal; in the first part of this section, we will discuss what properties $f \in \mathcal{S}$ must have to ensure that $\overline{\mathcal{O}(f)}$ and $(\overline{\mathcal{O}(f)}, Z^m)$ have the desired topological and dynamical properties. We will also determine for which elements of $\mathcal{S}$ the resulting evaluation cocycle will be covering and embedding. We will conclude this section with an example.

There is an extended real valued function on $\mathcal{S}$ given by

$$\| f \| = \sup\{|f(a+e_i) - f(a)| : a \in Z^m, 1 \le i \le m\}$$

for $f \in \mathcal{S}$. The importance of this function becomes apparent in the following lemma which is proved using Tychanoff's Theorem :

Lemma 2.1. *For $B_R = \{f \in \mathcal{S} : ||f|| \le R\} \subset \mathcal{S}$, B_R is a compact subset of $\mathcal{S}$ with respect to the topology of pointwise convergence on $\mathcal{S}$.*

When we construct the flow $(\overline{\mathcal{O}(f)}, Z^m)$ from an element f of $\mathcal{S}$, we want $\overline{\mathcal{O}(f)}$ to be a compact metric space. Because $||f|| = ||af||$ for all $f \in \mathcal{S}$ and $a \in Z^m$, this lemma suggests that it will be useful to insist that $||f|| < \infty$. Thus, we are led to the following definition:

Definition 2.1. *A map $\nu \in \mathcal{S}$ is called a net if*

$$||\nu|| = sup\{|\nu(a+e_i) - \nu(a)| : a \in Z^m, 1 \le i \le m\} < \infty \ .$$

Corollary 2.1. *If $\nu \in \mathcal{S}$ is a net then $\overline{\mathcal{O}(\nu)}$ is a compact metric space.*

Since we may also want the underlying Z^m flow which we construct to be minimal, we next ask what properties the net $\nu : Z^m \to R^m$ must have in order for $(\overline{\mathcal{O}(\nu)}, Z^m) = (X_\nu, Z^m)$ to be minimal.

Lemma 2.2. *For $\nu : Z^m \to R^m$ a net, (X_ν, Z^m) will be minimal if and only if ν is almost periodic. That is, (X_ν, Z^m) will be minimal if and only if given any finite collection $\{a_i\}_{i=1}^k \in Z^m$ and any $\epsilon > 0$, there exists a syndetic set $A \in Z^m$ such that if $a \in A$ then*

$$|a\nu(a_i) - \nu(a_i)| = |\nu(a + a_i) - (\nu(a) + \nu(a_i))| < \epsilon.$$

Proof. This follows immediately from the definition of almost periodic and from the fact that

$$\left\{\{x \in (R^m)^{Z^m} : |x(a_i) - r_i| \le \epsilon\}_{1 \le i \le k} : a_i \in Z^m, r_i \in R^m, \epsilon > 0, k \in N\right\}$$

forms a basis for the topology on $(R^m)^{Z^m}$. □

Visually this amounts to requiring that the pattern seen in the image of ν on a compact set will reoccur (up to ϵ) in the image of ν on Z^m with bounded gaps.

Thus, by using a net $\nu : Z^m \to R^m$ and the general procedure described here, we are able to construct a Z^m action on a compact set X_ν and a cocycle $h : X_\nu \times Z^m \to R^m$.

By insisting that ν is almost periodic we will also have (X_ν, Z^m) minimal. We now turn our attention to how the properties of the resulting evaluation cocycle depend on the net $\nu : Z^m \to R^m$.

Proposition 2.1. *Let $\nu : Z^m \to R^m$ be a net and let $h : X_\nu \times Z^m \to R^m$ be the evaluation cocycle associated with ν. Then h is embedding (covering) if and only if there exists κ, $\kappa' > 0$ such that*
$$\kappa|a-b| \le |\nu(a) - \nu(b)| \le \kappa'|a-b|$$
for all a, $b \in Z^m$ (for all a, $b \in Z^m$ with $|a-b|$ sufficiently large).

Proof. If h is embedding, then the existence of κ and κ' follows easily from Theorem 1.2.

The converse follows from the fact that for any $\mu = \lim b_i \nu \in X_\nu$ and any $a \in Z^m$, $|h(\mu, a)| = |\mu(a)| = \lim |b_i\nu(a)| = \lim |\nu(a+b_i) - \nu(b_i)|$.

A similar argument holds for the covering case. □

We conclude this section with an example, referred to as *the giant spiral.* Define $\nu_0 : R^2 \to R^2$ via

$$\nu_0(re^{i\theta}) = \begin{cases} re^{i\theta} & 0 \le r \le \exp(e^{2\pi}) \\ re^{i\theta}e^{i\log\log r} & \exp(e^{2\pi}) < r. \end{cases}$$

The following proposition demonstrates that ν_0 is a Lipschitz function from R^2 to R^2. (We'll denote by $\tilde{\nu_0}$ the net obtained by restricting ν_0 to the integer lattice; thus the following proposition demonstrates that $\tilde{\nu_0}$ is an embedding net.)

Proposition 2.2. *For $\nu_0 : R^2 \to R^2$ as defined above, there exists κ,κ' with $3/4 < \kappa < 1 < \kappa' < 5/4$ such that $\kappa|z-z'| \le |\nu_0(z) - \nu_0(z')| \le \kappa'|z-z'|$ for all $z, z' \in R^2$.*

The proof of this proposition and the following proposition, can be found in [M 94].

The following proposition says roughly that when $|z|$ is large, then, in a neighborhood of z, ν_0 is close to a rotation.

Proposition 2.3. *Let $B > 0$ and $\epsilon > 0$ be given. Then there exists $M > 0$ such that when $r \ge M$, then $|\nu_0(r'e^{i\theta'}) - r'e^{i\theta'}e^{i\log\log r}| < \epsilon$ for $r'e^{i\theta'} = re^{i\theta} + be^{i\beta}$, $0 \le \theta, \beta < 2\pi$ and $0 \le b \le B$.*

Theorem 2.1. *The spiral $\tilde{\nu_0} : Z^2 \to R^2$ has the following properties:*

1) *$X_{\tilde{\nu_0}} = \mathcal{O}(\tilde{\nu_0}) \cup \{R_\alpha : 0 \le \alpha < 2\pi\}$ where $R_\alpha : Z^2 \to R^2$ is rotation by α,*
2) *The evaluation cocycle $h_{\tilde{\nu_0}} : X_{\tilde{\nu_0}} \times Z^2 \to R^2$ is embedding but not close to linear.*

Proof. From Proposition 2.4, it is clear that
$$\mathcal{O}(\tilde{\nu_0}) \cup \{R_\alpha : 0 \le \alpha < 2\pi\} \subseteq X_{\tilde{\nu_0}}.$$
The proof of the converse is contained in [M 94].

The fact that $h_{\tilde{\nu_0}}$ is embedding follows from Proposition 2.1. Via Theorem 1.3, it is clear that $h_{\tilde{\nu_0}}$ is not close to linear. □

Because $(X_{\tilde{\nu_0}}, Z^2)$ has fixed points, it is not minimal. Thus the R^2 suspension $(X_{h_{\tilde{\nu_0}}}, R^2)$ of the flow $(X_{\tilde{\nu_0}}, Z^2)$ also will not be minimal. In fact, it is not difficult to verify that if $\pi(\mu, v) \in X_{h_{\tilde{\nu_0}}}$, then $\overline{\mathcal{O}\left(\pi(\mu, v)\right)} = X_{h_{\tilde{\nu_0}}}$ or $\overline{\mathcal{O}\left(\pi(\mu, v)\right)}$ is homeomorphic to the two dimensional torus [M 94].

We now ask whether we can construct an embedding cocycle which is not close to linear when the dynamics of (X, Z^2) are more complicated, say when (X, Z^2) is minimal. We will modify ν_0 so as to obtain an almost periodic net which still possesses many of the properties of ν_0. This modification is contained in Section 3.

3. The Almost Periodic Giant Spiral

Generally speaking, Lemma 2.2 tells us that a net $\mu : Z^2 \to R^2$ will be almost periodic if the pattern occurring in the image of μ on any disc centered at the origin reoccurs in the image of μ on R^2 with bounded gaps. So the basic idea behind the modification of ν_0 is as follows: we will start with a collection of discs of a certain size, one centered at the origin and the others appearing syndetically in R^2. The image of ν_0 will be unchanged on the complement of this collection of discs and on the disc centered at the origin. In the remaining discs, the pattern of the image of ν_0 on the disc centered at the origin is repeated and translated by the image of the center of the disc. That is, if $a \in Z^2$ is a center of a disc and if $a + be^{i\beta}$ is a point in the disc, then the image of $a + be^{i\beta}$ is $\nu_0(a) + \nu_0(be^{i\beta})$. This gives a new map $\nu_1 : R^2 \to R^2$ where the pattern of the image of ν_1 on a fixed size disc centered at the origin reoccurs in the image of ν_1 on R^2 with bounded gaps. We would then like to repeat the procedure; that is, we will take a collection of larger sized discs, one centered at the origin and the others appearing syndetically in R^2. The image of ν_1 will be unchanged on the complement of this collection of discs and on the larger disc centered at the origin. In the remaining larger discs, the pattern of the image of ν_1 on the larger disc centered at the origin is repeated and translated by the image of the center of the disc. We then repeat the procedure on even larger discs to obtain ν_3 and so on. The hope is that the limit of such a procedure will yield a map $\psi : R^2 \to R^2$ for which the pattern of the image of ψ on any sized disc about the origin occurs in the image of ψ on R^2 with bounded gap. That is, ψ is almost periodic.

In constructing ν_1, adjacent lattice points, one inside a disc and the other outside, could be separated by more than the original $||\tilde{\nu_0}||$. This means $||\nu_1|_{Z^2}||$ could be larger that $||\tilde{\nu_0}||$ and in general, $||\nu_i|_{Z^2}||$ could be larger than $||\nu_{i-1}|_{Z^2}||$. The limit of such a procedure may not result in a net of finite norm. This potential problem will be handled by insisting that at each stage the syndetically placed discs are surrounded by collars (that is, annuli whose inner radius is the radius of the discs). Each successive net ν_{i+1} will then be constructed as follows: the image of the previous map ν_i remains unchanged in the complement of the $(i+1)^{th}$ sized discs and collars and in the disc and collar centered at the origin. As before, the image of ν_i on the disc at the origin is repeated in the center of each of the syndetically placed discs. However, now the collar of each disc is used to gradually move from the pattern at the edge of the collar coming from ν_i to the pattern in the disc inside the collar. In this way, the norm of the resulting net $\nu_{i+1}|_{Z^2}$ will only increase by a small amount.

The modification of ν_0 that we have just described involves two steps; first, we need a procedure for constructing syndetic sets of varying sized discs with collars. In constructing these syndetic sets, we will need to keep in mind what properties they must have in order to facilitate defining the sequence of nets as described. A discussion of these desired properties and of the construction of the syndetic sets is contained in Section 3.1.

In the second step of the modification of the spiral, we will construct the sequence of nets as described. This will be done in Section 3.2.

3.1. **Syndetic Sets of Discs with Collars.** Let $0 < b_1 < B_1 < b_2 < B_2 < \cdots$ be an increasing sequence of real numbers. In the construction, b_i will be the radius of the interior of the i^{th} sized disc and $B_i - b_i$ the width of the i^{th} sized collar. We will require that for all $i \in N$, $120B_i \leq 2b_{i+1} \leq B_{i+1}$. That is, for $a \in Z^2$ the center of an i^{th} sized disc,

$$\begin{aligned} T_i(a) &= \{z \in R^2 : |z-a| \leq B_i\} \\ &= \{z \in R^2 : |z-a| \leq b_i\} \cup \{z \in R^2 : b_i < |z-a| \leq B_i\} \\ &= I_i(a) \cup C_i(a). \end{aligned}$$

We'll refer to $I_i(a)$ as the interior of the i^{th} sized disc centered at a and to $C_i(a)$ as the collar of the i^{th} sized disc centered at a. The union, $T_i(a)$, of $I_i(a)$ and $C_i(a)$ will be referred to as the i^{th} sized disc centered at a.

In the construction of the modified spiral, we will attempt to repeat the pattern in $I_i(0)$ in each of the interiors of the i^{th} sized discs, and we will be defining a new perturbation map in the collars. Thus, if $K_i \subseteq Z^2$ denotes the centers of the i^{th} sized discs, it is desirable that the sets $\{K_i\}_{i=1}^{\infty}$ have the following properties:

[SI] If a and a' are distinct elements of K_i, then $T_i(a) \cap T_i(a') = \emptyset$. (The i^{th} sized discs do not overlap.)

[SII] If $a \in K_i$, $a' \in K_j$ with $i < j$ then exactly one of the following holds:
(i) $T_i(a) \subseteq I_j(a')$ (ii)$T_i(a) \subseteq C_j(a')$ (iii)$T_i(a) \cap T_j(a') = \emptyset$.
(The i^{th} sized discs overlap with larger discs only in certain ways.)

[SIII] For $i < j$, $a' \in K_j$ then $K_i \cap I_j(a') = (K_i \cap I_j(0)) + a'$. (The pattern of occurrences of i^{th} sized discs in $I_j(a)$ is the same as the pattern in $I_j(0)$.)

In addition, of course, we will want each K_i to be syndetic. Provided $120B_i \leq 2b_{i+1} \leq B_{i+1}$, syndetic subsets $\{K_i\}_{i=1}^{\infty}$ of Z^2 with properties SI-SIII can be constructed; the complete proof of this can be found in [M 94]. In this section, we will give a general overview of the procedure used there.

Clearly it is easy to construct a syndetic set $K_1^1 \subseteq Z^2$ satisfying properties SI-SIII. For example, a sublattice will work. (The superscript is needed because this set will turn out to be our first approximation to K_1.) We then construct a set K_2^2 (the first approximation to K_2) which satisfies property SI. Again, a sublattice will do. The set K_1^1 may not overlap with K_2^2 in the right way so the elements of

K_1^1 may need to be moved so as to obtain properties SII and SIII. We'll denote this modification of K_1^1 by K_1^2. (K_1^2 is the second approximation of K_1 and in general, K_i^n is the $(n-i+1)^{th}$ approximation to K_i. i.e. K_i^n is the collection of centers of the i^{th} sized discs at the n^{th} stage.) We'll continue this process: At the n^{th} stage we will introduce the syndetic set K_n^n=$\{t_1k_ne_1 + t_2k_ne_2 : t_1, t_2 \in Z\}$ with $k_n \in N$ and $13B_n < k_n \leq 14B_n$. Then the sets $K_1^{n-1}, K_2^{n-1}, \dots, K_{n-1}^{n-1}$ must be modified to maintain the properties SI-SIII. (We'll require that these sets have properties SI-SIII only within b_n of the origin.) That is, the syndetic sets K_1^1 ,K_1^2, $K_2^2, \dots, K_1^n$, $K_2^n, \dots, K_n^n$ will have the following properties:

S(1): If a,a' are distinct elements of $K_i^n \cap I_n(0)$, then $d(T_i(a), T_i(a')) > B_i$,
S(2): If $a \in K_i^n \cap I_n(0)$, $a' \in K_j^n \cap I_n(0)$ for $1 \leq i < j \leq n$ then either
(i) $T_i(a) \subseteq I_j(a')$ and $\mathrm{d}(T_i(a), \partial I_j(a')) > B_i$ or
(ii) $T_i(a) \subseteq C_j(a')$ and $\mathrm{d}(T_i(a), \partial I_j(a')) > B_i$ or
(iii) $\mathrm{d}(T_i(a), T_j(a')) > B_i$,
S(3): For $i < j$ and $a' \in K_j^n \cap I_n(0)$, $K_i^n \cap I_j(a') = (K_i^n \cap I_j(0)) + a'$.

Properties S(1) and S(2) are stronger than the corresponding properties SI and SII. These added distance requirements will allow subsequent modifications of smaller discs to be made without introducing new, undesired overlaps.

K_i will be defined via

$$K_i = \bigcup_{n=i}^{\infty} \{K_i^n \cap \{z \in R^2 : |z| \leq b_n\}\} . \qquad (*)$$

Property S(4) ensures that this is well defined:

S(4): $K_i^n \cap I_{n-1}(0) = K_i^{n-1} \cap I_{n-1}(0)$ for $1 \leq n < m$.

Provided the sets $\{K_i^n\}_{n\geq i}$ have the same syndetic constant, K_i is syndetic [M 94]. For each $i \in N$ the constant k_i (used above in defining K_i^i) will describe the syndetic nature of K_i^n on $I_n(0)$:

S(5): $\{K_i^n \cap I_j(0)\} + \{z \in R^2 : |z| \leq 2k_i\} \supseteq I_j(0)$ for $1 \leq i < j \leq n$.

When moving an element $a \in Z^2$ of a set K_i^{n-1} to obtain properties S(1)-S(3), there are two modifications that will be allowed. First, suppose that an i^{th} sized disc $T_i(a)$ is within B_i of the boundary of the interior, $I_j(a')$ of a j^{th} sized disc:

1) By an *inner replacement* of a we mean replace a in K_i^{n-1} by $a + re^{i\theta} \in Z^2$ where $0 < r < 5B_i$ and $T_i(a + re^{i\theta}) \subseteq I_j(a')$ and $d(T_i(a + re^{i\theta}), \partial I_j(a')) > B_i$.
2) By an *outer replacement* of a, we mean replace a by $a + re^{i\theta} \in Z^2$ where $0 < r < 5B_i$ and $d(T_i(a + re^{i\theta}), I_j(a')) > B_i$.

These modifications will eliminate the undesired possibility of $T_i(a)$ straddling the boundary of the interior of a larger disc $T_j(a')$. We require the added distance to ensure that if an inner (or outer) replacement of a smaller disc, $T_l(a'')$, is subsequently necessary, this replacement will not move the disc $T_l(a'')$ into the i^{th} sized disc creating a new undesired overlap. (Recall $120B_l < B_i$.)

Modifications will also be necessary when a disc $T_i(a)$ is within B_i of the boundary of the collar of a larger disc, $T_j(a')$:

1) In a similar way, by an *outer replacement* of a we mean replace a by $a + re^{i\theta} \in Z^2$ where $0 < r < 5B_i$ and $\mathrm{d}(T_i(a + re^{i\theta}), T_j(a')) > B_i$.

We begin by describing when $n = 2$ how K_1^1 will be modified so as to obtain K_1^2: We have sublattices K_1^1 and K_2^2 as described. We may need to modify K_1^1 in $I_2(0)$ so as to obtain properties S(2) and S(3) so we consider any $a \in K_1^1 \cap I_2(0)$. If $\mathrm{d}(T_1(a), \partial I_2(0)) < B_1$, include an inner replacement of a in K_1^2. Otherwise, include a in K_1^2. Also include in K_1^2 any $a \in K_1^1 - I_2(0)$.

Since $K_1^2 \cap I_1(0) = \{0\} = K_1^1 \cap I_1(0)$, property S(4) holds. Property S(1) holds since discs about distinct elements of K_1^1 were at least $13B_1$ apart and thus modifications of magnitude at most $5B_1$ will not bring distinct elements of K_1^1 within B_1 of each other. Because $K_2^2 \cap I_2(0) = \{0\}$ properties S(2) and S(3) hold. Finally, property S(5) also holds as demonstrated in the following lemma:

Lemma 3.1. *For any $z \in I_2(0)$ there exists $a \in K_1^2 \cap I_2(0)$ such that $|z - a| \le 2k_1$.*

Proof. Let $z \in I_2(0)$. By definition of K_1^1, there exists $a \in K_1^1 \cap I_2(0)$ such that $|a - z| < k_1$. If $a \in K_1^2$, we're done. Otherwise a was inner replaced to yield $a' \in K_1^2 \cap I_2(0)$ with $|a - a'| \le 5B_1 < k_1$. Thus, $|z - a'| \le |z - a| + |a - a'| \le 2k_1$ as desired. □

To construct $K_1^1, K_1^2, K_1^3, K_2^3, \ldots, K_1^n, \ldots, K_{n-1}^n$ we will use induction. After relabeling, the argument just described for $m = 1$ will enable us to construct $K_m^{m+1} \subseteq Z^2$ so that the sets

$$\begin{array}{l} K_m^m \\ K_m^{m+1}, K_{m+1}^{m+1} \end{array}$$

have properties S(1)-S(5) for any $m \in N$. Now suppose whenever we have sets on the left below with properties S(1)-S(5) we can construct the subsets $K_m^{m+n-1}, \ldots, K_{m+n-2}^{m+n-1}$ of Z^2 so that the sets on the right below have properties S(1)-S(5):

$$\begin{array}{lll} K_m^m & \qquad & K_m^m \\ K_m^{m+1}, K_{m+1}^{m+1} & & K_m^{m+1}, K_{m+1}^{m+1} \\ \vdots \quad \vdots \quad \vdots & & \vdots \quad \vdots \quad \vdots \\ K_m^{m+n-2}, \ldots, K_{m+n-2}^{m+n-2} & & K_m^{m+n-2}, \ldots, K_{m+n-2}^{m+n-2} \\ & & K_m^{m+n-1}, \ldots, K_{m+n-2}^{m+n-1}, K_{m+n-1}^{m+n-1} \end{array}$$

This is our induction hypothesis. We would now like to show that we can do a similar construction for n different sizes of discs. In fact, we will do this construction for $m = 1$. The same argument, after relabeling, yields the more general result. So suppose that the sets

$$\begin{array}{l} K_1^1 \\ K_1^2, K_2^2 \\ \vdots \quad \vdots \quad \vdots \\ K_1^{n-1}, \ldots, K_{n-1}^{n-1} \end{array}$$

have properties S(1)-S(5). Then the sets on the left below also have properties S(1)-S(5) and we may use our induction hypothesis to construct subsets $K_2^n, \dots, K_n^n$ of Z^2 so that the sets on the right below have properties S(1)-S(5).

$$\begin{array}{ll} K_2^2 & K_2^2 \\ K_2^3, K_3^3 & K_2^3, K_3^3 \\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots \quad \vdots \\ K_2^{n-1}, \dots, K_{n-1}^{n-1} & K_2^{n-1}, \dots, K_{n-1}^{n-1} \\ & K_2^n \;, \dots, \; K_{n-1}^n, K_n^n \end{array}$$

Thus, we have the following collection of sets with the properties S(1)-S(5):

$$\begin{array}{l} K_1^1 \\ K_1^2, K_2^2 \\ \vdots \quad \vdots \quad \vdots \\ K_1^{n-1}, \dots, K_{n-1}^{n-1} \\ \quad K_2^n, ..., K_{n-1}^n, K_n^n. \end{array}$$

So it remains to show that we can construct K_1^n.

We proceed as follows: First, suppose $a \in K_1^{n-1} \cap I_{n-1}(0)$. In order to satisfy property S(4), include a in K_1^n.

Next consider $a \in K_1^1 \cap I_n(0)$. The elements of the sets $K_2^n, \dots, K_n^n$ satisfy properties S(1)-S(5). Also, if $a, a' \in K_1^1 \cap I_n(0)$ with $a \neq a'$, then $|a - a'| \geq 13B_1$. Thus since $120B_i \leq 2b_{i+1} \leq B_{i+1}$, for each element of $K_1^1 \cap I_n(0)$, exactly one of the following can occur. We assume, as the list progresses, that if an element is in one of the cases, it is not in any of the previous cases. Along with each case is the appropriate constructive step.

- If $0 \leq d(T_1(a), \partial I_n(0)) \leq B_1$, do an inner replacement of a.
- If $\inf\{d(T_1(a), T_i(a')) : a' \in K_i^n \cap I_n(0), 2 \leq i \leq n-1\} > B_1$, include a in K_1^n.
- If $T_1(a) \subseteq I_i(a')$ for some $a' \in K_i^n \cap I_n(0)$ and $2 \leq i \leq n-1$, discard a.
- If $T_1(a) \not\subseteq C_i(a')$ but $d(T_1(a), \partial T_i(a')) \leq B_1$ for some $a' \in K_i^n \cap I_n(0)$ and $2 \leq i \leq n-1$, do an outer replacement of a.
- If $d(T_1(a), \partial I_i(a')) \leq B_1$ for some $a' \in K_i^n \cap I_n(0)$ and $2 \leq i \leq n-1$, do an outer replacement of a.
- If $T_1(a) \subseteq C_i(a')$ and $d(T_1(a), \partial T_i(a')) > B_1$ for some $a' \in K_i^n \cap I_n(0)$ and $2 \leq i \leq n-1$, include a in K_1^n.

To complete K_1^n, we also add the following two collections of points to K_1^n: first add

$$\bigcup_{i=2}^{n} \bigcup_{a \in K_i^n \cap I_n(0)} (a + \{K_1^n \cap I_i(0)\})$$

and secondly, add $\{a \in K_1^1 - I_n(0)\}$. Clearly, this will give us property S(3). By construction, K_1^n also satisfies properties S(1) (since as previously discussed, the elements of K_1^1 are sufficiently far apart that the discs about two elements of K_1^1 remain at a distance of at least B_1 even after replacements) and S(2). We note that K_1^n will be syndetic since K_1^1 was and we only modified finitely many elements of K_1^1 in constructing K_1^n. It remains to show only that K_1^n has property S(5); this is done in [M 94] and, although there are several cases to be checked, the flavor of the argument found there is similar to the proof of Lemma 3.1.

We define

$$K_n = \bigcup_{m \geq n} \{K_n^m \cap I_m(0)\};$$

K_n is syndetic with $K_n + \{z \in R^2 : |z| \leq 2k_n\} = R^2$. The collection of sets $\{K_n\}_{n=1}^{\infty}$ has the properties SI-SIII. In fact it has the stronger properties S(1)-S(5).

The described procedure for constructing syndetic sets of varying sized discs required only that $120B_i \leq 2b_{i+1} \leq B_{i+1}$. It remains to specify the size of the inner portions and collars we will use in the construction of the modified spiral and to observe some properties of these inner portions and collars. Of course, our choice of sizes will be motivated by the properties of the spiral, ν_0, especially Proposition 2.3.

We now choose the sequence $0 < b_1 < B_1 < b_2 < B_2 < \cdots$ as follows: Set $b_1 = \exp(e^{4\pi})$ and $B_1 = \exp(e^{8\pi})$. This will be the radius of the inner portion and collar of the first sized disc used in the construction of the modified spiral. Choose M_1 so that $M_1 > 13B_1$ and so that it satisfies Proposition 2.3 for $B = B_1$ and $\epsilon = \frac{1}{4^1}$. (Thus if $|a| > M_1$, ν_0 is within $\frac{1}{4}$ of a rotation on a disc of radius B_1 centered at a.)

Continue in the same way; choose $k_n \in N$ so that $k_n > k_{n-1} + 2$ and $b_n = \exp(e^{2k_n\pi}) > M_{n-1}$. (Thus if $|a| > b_n$, ν_0 is within $\frac{1}{4^{n-1}}$ of a rotation on a disc of radius B_{n-1} centered at a.) Set $B_n = \exp(e^{2(k_n+2)\pi})$ and choose M_n so that $M_n > 13B_n$ and so that it satisfies Proposition 2.3 for $B = B_n$ and $\epsilon = \frac{1}{4^n}$.

Notice that if $|z| = b_n$ or $|z| = B_n$ then $\nu_0(z) = z$ and that two full rotations occur as $|z|$ increases from b_n to B_n. We will denote by β_n the radius of the circle in the collar, $C_n(0)$, which is fixed by ν_0; so $\beta_n = \exp(e^{2(k_n+1)\pi})$.

Observation 3.1. *For any $n \in N$,*

1) $B_n \geq 4^{n+2}\beta_n \geq 4^{n+2}b_n \geq b_n \geq 4^{n+2}B_{n-1}$,
2) $B_n - \beta_n \geq 4^{n+1}\beta_n$,
3) $B_n - b_n \geq 4^{n+1}b_n$,
4) $B_n - B_{n-1} \geq 4^{n+1}B_{n-1}$.

(As a result of this observation, we see that the sizes $\{b_n, B_n\}_{n\in N}$ so defined satisfy the requirement $120B_i \leq 2b_{i+1} \leq B_{i+1}$.)

In the construction of the sets $K_1^1, K_1^2, K_2^2, \ldots, K_1^n, \ldots, K_n^n$ we will begin with $K_n^n = \{t_1k_ne_1 + t_2k_ne_2 : t_1, t_2 \in Z\}$ where $k_n \in N$ and $4^nM_n < k_n \leq 2(4^n)M_n$.

We conclude this section with several useful observations:

Observation 3.2. *Let a, $a' \in K_i^n$ with $a \neq a'$ and $i \leq n$. If $z \in T_i(a)$ and $z' \in T_i(a')$ then*

$$|z - z'| \geq 4^{i+1}B_i$$

Observation 3.3. *Suppose* $0 \le b \le b_a$ *where* $b_a \in (b_i, \beta_i]$. *Then*
$$|\nu_0(be^{i\theta}) - be^{i\theta}e^{i\log\log b_a}| \le \frac{1}{4^i}(b_a - b).$$

3.2. **The Modification of the Spiral.** In the previous subsection, we described a procedure for placing collections of discs with collars of varying sizes syndetically in R^2. In the second step of the modification of the spiral we will construct the sequence of nets whose general description was given at the beginning of the section. The basic idea behind the construction of this sequence of nets involved modifying the spiral on the discs in R^2; we begin with two propositions which describe in general how to modify an embedding net on some subset of the plane. The proofs of these propositions are straight forward and are found in [M 94].

The first of these propositions describes how to modify a net on a subset of the plane and still end up with a net.

Proposition 3.1. *Suppose* $\mu : R^2 \to R^2$ *is a net with* $||\mu|| \le M$. *Let* $S \subseteq R^2$ *and* $\nu : S \to R^2$ *such that*

1. *If* $0 \in S$, *then* $\nu(0) = 0$,
2. $\sup\{|\nu(z) - \nu(z+e_i)| : z, z+e_i \in S, i = 1,2\} \le M'$,
3. *There exists* $\epsilon > 0$ *such that when* $z \in S$ *with* $d(z, \partial(S)) \le 1$ *then* $|\nu(z) - \mu(z)| < \epsilon$.

Define $\psi : R^2 \to R^2$ *by*
$$\psi(z) = \begin{cases} \nu(z) & z \in S \\ \mu(z) & \text{otherwise.} \end{cases}$$
Then ψ *is a net with* $||\psi|| \le \max\{M', M + \epsilon\}$.

The issue of maintaining the embedding property after a modification is addressed in the next proposition. Hypothesis (3) may seem a bit abstruse. In the case of the almost periodic spiral, an intuitive description of how this hypothesis will be satisfied is as follows: The subset of the plane to be modified will consist of discs with collars. If a point is in the inner portion of one of these discs then it is "far" from the boundary of the disc and "close" to a center. The center is well behaved with respect to the background net. Otherwise, if a point is in a collar and "close" to the boundary of the disc, then the image under the net in the collar will be close to the image under the original net.

Proposition 3.2. *Suppose* $\mu : R^2 \to R^2$ *is a net with* $m|z - z'| \le |\mu(z) - \mu(z')|$ *whenever* $|z - z'| \ge 1$ *for some* $0 < m < 1$. *Let* $S \subseteq R^2$ *and* $\nu : S \to R^2$ *such that*

1. ν *satisfies the hypotheses of Proposition 3.1,*
2. *When* $z, z' \in S$ *and* $|z - z'| \ge 1$, *then* $m'|z - z'| \le |\nu(z) - \nu(z')|$ *for some* $0 < m' < 1$,
3. *There exists* $0 < \epsilon < m$ *such that for all* $z \in S$ *either*
   (i) $|\nu(z) - \mu(z)| \le \epsilon$ *or*
   (ii) There exists $R > 1$ *and* $a \in S$ *such that* $d(z, \partial S) \ge \frac{2R}{\epsilon}$, $|z - a| \le R$, $|\nu(z) - \nu(a)| \le R$ *and* $\nu(a) = \mu(a)$.

Form $\psi : R^2 \to R^2$ *via*
$$\psi(z) = \begin{cases} \nu(z) & z \in S \\ \mu(z) & \text{otherwise.} \end{cases}$$

Then ψ *is a net and, when* $|z-z'| \geq 1$, $K|z-z'| \leq |\psi(z)-\psi(z')|$ *where* $K = \min\{m-\epsilon, m'\}$.

Recall that the basic idea behind the modification of ν_0 is the following: for each $n \in N$ we will construct a net, ν_n, which is "almost periodic" with respect to n different sizes of discs which occur syndetically in R^2. Starting with the spiral ν_0, we will construct this collection of nets satisfying:

NI: For $1 \leq i \leq n$, $a \in K_i^n \cap I_n(0)$, $0 \leq b \leq b_i$, $0 \leq \theta < 2\pi$ we have
$$\nu_n(a+be^{i\theta}) = \nu_n(a) + \nu_n(be^{i\theta}),$$
NII: For any $z, z' \in R^2$, $|\nu_n(z) - \nu_n(z')| \leq M|z-z'|$ and when $|z-z'| \geq 1$, $m|z-z'| \leq |\nu_n(z)-\nu_n(z')|$ where m, $M > 0$ are independent of n,
NIII: $\nu_n = \nu_{n-1}$ on $I_{n-1}(0)$.

We then define $\psi : R^2 \to R^2$ by
$$\psi(z) = \nu_n(z) \qquad \text{when} \qquad |z| \leq b_n.$$
By NIII above, ψ is well defined. The following proposition follows directly from NI-NIII and Propositions 3.1 and 3.2:

Proposition 3.3. *For* $\psi : R^2 \to R^2$ *as defined above,* ψ *is an embedding, almost periodic net.*

We begin by constructing a more general type of net $\mu_{i,i+k}^n : R^2 \to R^2$, for all $n \in N$ and all i, k with $1 \leq i \leq n$ and $0 \leq k \leq n-i$. (The indices associated with this net indicate that it has the desired "almost periodic like" behavior (described in N(1) below) on $I_n(0)$ for the syndetic sets $K_i^n, K_{i+1}^n, \ldots, K_{i+k}^n$.) The nets $\{\mu_{i,i+k}^n : n \in N,\ 1 \leq i \leq n,\ 0 \leq k \leq n-i\}$ will have the following properties:

N(1): For any $a \in K_{i+j}^n \cap I_n(0)$, $0 \leq j \leq k$ and $be^{i\theta}$ with $0 \leq b \leq b_{i+j}$, $0 \leq \theta < 2\pi$ we have $\mu_{i,i+k}^n(a+be^{i\theta}) = \mu_{i,i+k}^n(a) + \mu_{i,i+k}^n(be^{i\theta})$,

N(2): For any $z, z' \in R^2$ with $|z-z'| \geq 1$,
$$\left(\kappa - \sum_{j=0}^k \left(\tfrac{1}{4^{i+j}}\right)\right)|z-z'| \leq |\mu_{i,i+k}^n(z) - \mu_{i,i+k}^n(z')| \leq \left(\kappa' + \sum_{j=0}^k \left(\tfrac{1}{4^{i+j}}\right)\right)|z-z'|,$$
(Recall κ and κ' were the constants associated with ν_0 from Proposition 2.2.)

N(3): $\mu_{i,i+k}^n = \mu_{i,i+k}^{n-1}$ on $I_{n-1}(0)$ provided $i+k \leq n-1$,

N(4): $\mu_{i,i+k}^n = \mu_{i,i+k-1}^n$ on $I_{i+k}(0)$ provided $1 \leq k \leq n-i$,

N(5): In $I_n(0)$, $\mu_{i,i+k}^n$ and $\mu_{i+1,i+k}^n$ agree on all but the punctured i^{th} sized discs (Notice that they do agree at the centers of the i^{th} sized discs.),

N(6): In $I_n(0)$, $\mu_{i,i+k}^n = \nu_0$ off the i to $i+k$ sized discs. In addition, if $a \in I_n(0)$ is the center of a j^{th} sized disc, for $i \leq j \leq i+k$, and if $T_j(a)$ is not contained in any larger discs, then $\mu_{i,i+k}^n(a) = \nu_0(a)$ as well,

N(7): Suppose $a \in K_{i+j}^n \cap I_n(0)$ for $0 \leq j \leq k$ and $z = a + be^{i\theta}$ is in $C_{i+j}(a)$ but is not a non-center point in any smaller discs. Then
$$\mu_{i,i+k}^n(z) = \begin{cases} \mu_{i,i+k}^n(a) + \nu_0(be^{i\theta}) & b_{i+j} \leq b \leq b_a \\ \mu_{i,i+k}^n(a) + be^{i\theta}e^{i\log\log b_a} & b_a < b \leq B_{i+j} \end{cases}$$

for some $b_a \in (b_i, \beta_i]$.

Assume $\mu^n_{i,i+k}$ can be constructed with the properties described above for all $n \in N$ and $1 \le i \le n$, $0 \le k \le n-i$. Then, by setting $\nu_n = \mu^n_{1,n}$, we will have a collection of nets which satisfy NI, NII and NIII. Clearly NI holds because of the "almost periodic like" behavior described in N(1) that the nets $\mu^n_{1,n}$ would have, and from N(2) we see that NII holds. Property NIII will also hold; by N(3), with $i = 1$ and $k = n-2$, we see $\mu^n_{1,n-1} = \mu^{n-1}_{1,n-1}$ on $I_{n-1}(0)$ and by N(4), with $i = 1$ and $k = n-1$, we see $\mu^n_{1,n} = \mu^n_{1,n-1}$ on $I_n(0)$ which contains $I_{n-1}(0)$. Properties N(5), N(6) and N(7) will help us to better understand the behavior of the nets we have constructed at each stage.

We will now show that we can construct $\mu^n_{i,i+k}$ for all $n \in N$ and all i, k with $1 \le i \le n$ and $0 \le k \le n-i$. The proof is by induction on k so we will first construct $\mu^n_{i,i}$.

Let $S = \bigcup \{T_i(a) - \{a\} : a \in K^n_i \cap I_n(0), a \ne 0\}$. Let $\nu : S \to R^2$ via, for $z = a + be^{i\theta} \in S$, $a \in K^n_i \cap I_n(0)$ and $0 \le b \le B_i$

$$\nu(z) = \begin{cases} \nu_0(a) + \nu_0(be^{i\theta}) & 0 \le b \le b_a \\ \nu_0(a) + be^{i\theta}e^{i\log\log b_a} & b_a < b \le B_i \end{cases}$$

where $b_a \in (b_i, \beta_i]$ is such that $\nu_0(a) = ae^{i\log\log b_a}$ (i.e. $e^{i\log\log|a|} = e^{i\log\log b_a}$). So $e^{i\log\log b_a}$ is the rotation felt by a about the origin under the net ν_0. The image under ν of the annulus $\{z \in R^2 : b_a < |z-a| \le B_i\}$ is just the annulus rotated by $e^{i\log\log b_a}$.

Now define $\mu^n_{i,i} : R^2 \to R^2$ via

$$\mu^n_{i,i}(z) = \begin{cases} \nu(z) & z \in S \\ \nu_0(z) & z \notin S \end{cases}$$

Clearly $\mu^n_{i,i}$ has properties N(1), N(6) and N(7) by construction. It has property N(3) since $K^n_i = K^{n-1}_i$ on $I_{n-1}(0)$. Properties N(4) and N(5) do not apply when $k = 0$ since $\mu^n_{i,i-1}$ is not defined. The fact that it has property N(2) follows from the following two lemmas.

Lemma 3.2. *For $\mu^n_{i,i}$ as defined above, $||\mu^n_{i,i}|| \le \left(\kappa' + \frac{1}{4^i}\right)$.*

Proof. We will verify that ν_0 and ν satisfy the hypotheses of Proposition 3.1. Hypothesis 1 is clear.

If z, $z + e_i \in S$ then z, $z + e_i \in T_i(a)$ for some $a \in K^n_i \cap I_n(0)$ (This follows from Observation 3.2.). Thus,

$$\sup\{|\nu(z) - \nu(z+e_i)| : z, z+e_i \in S,\ i = 1,2\} \le ||\nu_0|| \le \kappa'$$

and hypothesis 2 is satisfied.

To check hypothesis 3, consider $z \in S$ where $z = a + be^{i\theta} \in T_i(a)$ for some $a \in K^n_i \cap I_n(0)$, $a \ne 0$ and $b_a \le b \le B_i$. Recall $|a| > M_i$ and $M_i > 0$ was chosen as in Proposition 2.3 for $B = B_i$ and $\epsilon = \frac{1}{4^i}$. Thus

$$|\nu_0(z) - \nu(z)| = |\nu_0(a + be^{i\theta}) - (a + be^{i\theta})e^{i\log\log b_a}| < \tfrac{1}{4^i}.$$

In particular this holds for $z \in S$ with $d(z, \partial S) \le 1$. □

Lemma 3.3. *For all $z, z' \in R^2$ with $|z - z'| \geq 1$,*

$$\left(\kappa - \frac{1}{4^i}\right)|z - z'| \leq |\mu^n_{i,i}(z) - \mu^n_{i,i}(z')|.$$

Proof. We will show that ν_0 and ν satisfy the hypotheses of Proposition 3.2. Hypothesis 1 is covered in the proof of the previous lemma. We turn our attention next to hypothesis 3: let $z \in S$. So $z = a + be^{i\theta} \in T_i(a)$ for some $a \in K^n_i \cap I_n(0)$, $a \neq 0$ and $0 \leq b \leq B_i$. As in the proof of the previous lemma, when $b_a \leq b \leq B_i$, $|\nu_0(z) - \nu(z)| < \frac{1}{4^i}$. Otherwise, $b < b_a \leq \beta_i$ and

- using Observation 3.1, $d(z, \partial S) = d(z, \partial T_i(a)) \geq B_i - \beta_i \geq 4^{i+1}\beta_i$,
- $|z - a| < \beta_i$ with $\nu_0(a) = \nu(a)$ and $|\nu(a) - \nu(z)| < \beta_i$.

So with $R = \beta_i$ and $\epsilon = \frac{1}{4^i}$, we see hypothesis 3 is satisfied.

Finally, we verify that ν satisfies hypothesis 2 of Proposition 3.2. So let $z, z' \in S$. There are two possibilities: either $z \in T_i(a)$, $z' \in T_i(a')$ with $a \neq a'$ or $z, z' \in T_i(a)$. Suppose first that $z = a + be^{i\theta} \in T_i(a)$ and $z' = a' + b'e^{i\theta'} \in T_i(a')$ with $a \neq a'$, $a, a' \in (K^n_i \cap I_n(0)) - \{0\}$. We notice $|\nu_0(a) - \nu_0(a')| \geq \kappa|a - a'|$ by Proposition 2.2 and also, by the definition of ν, $|\nu_0(a) - \nu(z)| \leq B_i$ (similarly for $a'$and z'). Thus

$$\begin{aligned} |\nu(z) - \nu(z')| &\geq \big||\nu_0(a) - \nu_0(a')| - |\nu(z) - \nu_0(a) - \nu(z') + \nu_0(a')|\big| \\ &\geq \kappa|a - a'| - 2B_i \geq \kappa|z - z'| - (2 + 2\kappa)B_i \\ &\geq \kappa|z - z'| - 4B_i \geq \left(\kappa - \tfrac{1}{4^i}\right)|z - z'|. \end{aligned}$$

So hypothesis 2 of Proposition 3.2 is satisfied when z, z' fall into the first case. Otherwise, $z = a + be^{i\theta} \in T_i(a)$ and $z' = a + b'e^{i\theta'} \in T_i(a)$ with $a \in (K^n_i \cap I_n(0)) - \{0\}$. There are three possibilities for b and b'. Suppose first that $b, b' \leq b_a$. Then

$$|\nu(z) - \nu(z')| = |\nu_0(be^{i\theta}) - \nu_0(b'e^{i\theta'})| \geq \kappa|be^{i\theta} - b'e^{i\theta'}| = \kappa|z - z'|.$$

Next suppose that $b, b' \geq b_a$. Then

$$|\nu(z) - \nu(z')| = |be^{i\theta}e^{i\log\log b_a} - b'e^{i\theta'}e^{i\log\log b_a}| \geq |be^{i\theta} - b'e^{i\theta'}| = |z - z'|.$$

The last possibility is, without loss of generality, that $b < b_a \leq b'$. In this case,

$$\begin{aligned} |\nu(z') - \nu(z)| &= |b'e^{i\theta'}e^{i\log\log b_a} - \nu_0(be^{i\theta})| \\ &\geq \big||b'e^{i\theta'}e^{i\log\log b_a} - be^{i\theta}e^{i\log\log b_a}| - |be^{i\theta}e^{i\log\log b_a} - \nu_0(be^{i\theta})|\big| \\ &\geq \big||b'e^{i\theta'} - be^{i\theta}| - |be^{i\theta}e^{i\log\log b_a} - \nu_0(be^{i\theta})|\big|. \end{aligned}$$

But by Observation 3.3,

$$|be^{i\theta}e^{i\log\log b_a} - \nu_0(be^{i\theta})| \leq \tfrac{1}{4^i}(b_a - b) \leq \tfrac{1}{4^i}|z - z'|$$

and $|b'e^{i\theta'} - be^{i\theta}| = |z - z'|$. Thus,

$$|\nu(z) - \nu(z')| \geq |z - z'| - \tfrac{1}{4^i}|z - z'| \geq \left(\kappa - \tfrac{1}{4^i}\right)|z - z'|.$$

□

Thus, for $k = 0$, $\mu^n_{i,i+k}$ having properties N(1)-N(7) can be constructed. Now suppose that for a fixed $k > 0$, we can construct a net $\mu^n_{i,i+k}$ with properties N(1)-N(7) for any $1 \leq i < n$ for which $i + k \leq n$. This is our induction hypothesis; that is, we assume we are able to construct a net which has the desired "almost periodic like" behavior in $I_n(0)$ with respect to $k + 1$ different sizes of discs. We will show that, when $i + k + 1 \leq n$, $\mu^n_{i,i+k+1}$ having properties N(1)-N(7) can be constructed as well.

Use the induction hypothesis to construct $\mu^n_{i,i+k}, \mu^n_{i+1,i+1+k} : R^2 \to R^2$. Notice that $\mu^n_{i,i+k}$ has the desired 'almost periodic like" behavior in $I_n(0)$ with respect to the $k+1$ discs with centers in $K^n_{i+j} \cap I_n(0)$ for $0 \le j \le k$ while $\mu^n_{i+1,i+1+k}$ recognizes discs with centers in $K^n_{i+j} \cap I_n(0)$ for $1 \le j \le k+1$. The net $\mu^n_{i,i+k}$ recognizes all the discs we want $\mu^n_{i,i+k+1}$ to recognize except those with centers in $K^n_{i+k+1} \cap I_n(0)$ while the net $\mu^n_{i+1,i+1+k}$ recognizes all the discs we want $\mu^n_{i,i+k+1}$ to recognize except those with centers in $K^n_i \cap I_n(0)$. So in constructing $\mu^n_{i,i+k+1}$, we will use $\mu^n_{i+1,i+1+k}$ as the background net. Then it is only the discs with centers in $K^n_i \cap I_n(0)$ that will need to be modified. As it turns out, it will be more convenient to modify $\mu^n_{i+1,i+1+k}$ on the following subset of R^2 (which certainly contains all the discs with centers in $K^n_i \cap I_n(0)$): Let

$$S = \left(\bigcup_{j=i+1}^{i+1+k} \bigcup_{a \in K^n_j \cap I_n(0)} I_j(a) \right) \bigcup \left(\bigcup_{a \in K^n_i \cap I_n(0)} T_i(a) \right).$$

We observe that if $z \in S$ then it falls into one of three possible cases.

Case 1: z is in the inner portion of one of the larger discs. That is, $z = a' + b'e^{i\theta'} \in I_j(a')$ where $a' \in K^n_j$, $0 \le b' \le b_j$. *If z falls into this case we always assume that j is the largest index for which this is true.*

Case 2: z is in an i^{th} sized disc which is in the collar of a larger j^{th} sized disc. That is, $z = a + be^{i\theta} = a' + re^{i\theta} + be^{i\theta} = a' + b'e^{i\theta'} \in T_i(a) \subset C_j(a')$, $0 \le b \le B_i$ and $b_j \le r, b' \le B_j$. *If z falls in to this case, we always make two assumptions. First we assume that z does not fall into case 1. Secondly, we assume that j is the smallest index for which this is true.*

Case 3: z is in an i^{th} sized disc which does not intersect any of the larger discs. That is, $z = a + be^{i\theta}$ where $a \in K^n_i$, $0 \le b \le B_i$.

We now define $\nu : S \to R^2$ as follows:

$$\nu(z) = \begin{cases} \mu^n_{i+1,i+1+k}(a') + \mu^n_{i,i+k}(b'e^{i\theta'}) & z \text{ in Case 1} \\ \begin{cases} \mu^n_{i+1,i+1+k}(a) + \mu^n_{i,i+k}(be^{i\theta}) & 0 \le b \le b_a \\ \mu^n_{i+1,i+1+k}(a) + be^{i\theta}e^{i\log\log b_a} & b_a < b \le B_i \end{cases} & z \text{ in Case 2} \\ \begin{cases} \mu^n_{i+1,i+1+k}(a) + \mu^n_{i,i+k}(be^{i\theta}) & 0 \le b \le b_a \\ \mu^n_{i+1,i+1+k}(a) + be^{i\theta}e^{i\log\log b_a} & b_a < b \le B_i \end{cases} & z \text{ in Case 3} \end{cases}$$

(We'll refer to this as EQ 3.1.) The difference between cases 2 and 3 above is how the value b_a is obtained. If z is in case 3, then, as before, $b_a \in (b_i, \beta_i]$ is such that $e^{i\log\log b_a}$ is the rotation felt by a about the origin under the net ν_0. (By property N(6) of $\mu^n_{i+1,i+1+k}$, we know $\nu_0(a) = \mu^n_{i+1,i+1+k}(a) = ae^{i\log\log b_a}$).

If z is in case 2, then $z \in T_i(a) \subset C_j(a')$. Since j the smallest index for which this is true and since z does not fall into case 1, a is not in any of the $i+1$ through $j-1$ sized discs. So $a = a' + re^{i\theta} \in C_j(a')$ and, by property N(7) of $\mu^n_{i+1,i+1+k}$,

$$\mu^n_{i+1,i+1+k}(a) = \begin{cases} \mu^n_{i+1,i+1+k}(a') + \nu_0(re^{i\theta}) & b_j \le r \le b_{a'} \\ \mu^n_{i+1,i+1+k}(a') + re^{i\theta}e^{i\log\log b_{a'}} & b_{a'} \le r \le B_j. \end{cases}$$

for some $b_{a'} \in (b_j, \beta_j]$. In this case, choose $b_a \in (b_i, \beta_i]$ such that $e^{i \log\log b_a} = e^{i\gamma}$ where $\mu^n_{i+1,i+1+k}(a) = \mu^n_{i+1,i+1+k}(a') + re^{i\theta}e^{i\gamma}$. So $e^{i\log\log b_a} = e^{i\log\log r}$ when $r < b_{a'}$ and $e^{i\log\log b_a} = e^{i\log\log b_{a'}}$ when $r \geq b_{a'}$; $e^{i\log\log b_a}$ is the rotation felt by a about a' under the net $\mu^n_{i+1,i+1+k}$.

Now define $\mu^n_{i,i+k+1} : R^2 \to R^2$ by (EQ 3.2):

$$\mu^n_{i,i+k+1}(z) = \begin{cases} \nu(z) & z \in S \\ \mu^n_{i+1,i+1+k}(z) & z \notin S. \end{cases}$$

Having constructed the maps $\mu^n_{i,i+k+1}$ we must now show that they satisfy the required properties N(1)-N(7). The following proposition shows that properties N(1) and N(3)-N(7) are satisfied. The fact that $\mu^n_{i,i+k+1}$ satisfies property N(2) is more difficult to verify and its proof is omitted (See [M 94]). (The flavor of the argument and many of the techniques used are similar to those in the proofs of Lemmas 3.2 and 3.3; by considering various cases, it is verified that $\mu^n_{i+1,i+k+1}$ and ν satisfy the hypotheses of Propositions 3.2 and 3.3.)

Proposition 3.4. *The map $\mu^n_{i,i+k+1}$ so defined has properties N(1) and N(3)-N(7).*

Proof. Property N(3): Since $K^n_j = K^{n-1}_j$ on $I_{n-1}(0)$ (from property S(4) of the constructed syndetic sets), property N(3) follows.

Property N(4): Let $z \in I_{i+k+1}(0)$. Then $z \in S$ is in case 1 and from EQ. 3.2 and 3.1, $\mu^n_{i,i+k+1}(z) = \nu(z) = \mu^n_{i+1,i+k+1}(0) + \mu^n_{i,i+k}(z) = \mu^n_{i,i+k}(z)$ as desired.

Property N(5): Let $z \in I_n(0) - \bigcup\{T_i(a) - \{a\} \ : \ a \in K^n_i \text{ and } a \neq 0\}$. If $z \notin S$, then $\mu^n_{i,i+k+1}(z) = \mu^n_{i+1,i+k+1}(z)$ (EQ 3.2) and we're done. Otherwise, $z \in S$ and z must fall into case 1. Say $z = a' + b'e^{i\theta'}$ for $a' \in K^n_j$ where $i + 1 \leq j \leq i + k + 1$. Notice by property S(3) of the constructed syndetic sets, $b'e^{i\theta'} \in I_j(0) - \cup\{T_i(a) - \{a\} \ : \ a \in K^n_i\}$. Using induction hypothesis N(5) on $\mu^n_{i,i+k}$ we see $\mu^n_{i,i+k}(b'e^{i\theta'}) = \mu^n_{i+1,i+k}(b'e^{i\theta'})$. Also, since $I_j(0) \subseteq I_{i+k+1}(0)$, using induction hypothesis N(4) on $\mu^n_{i+1,i+k}$ we see $\mu^n_{i+1,i+k}(b'e^{i\theta'}) = \mu^n_{i+1,i+k+1}(b'e^{i\theta'})$. So, using EQ. 3.2 and 3.1,

$$\begin{aligned}\mu^n_{i,i+k+1}(z) = \nu(z) &= \mu^n_{i+1,i+k+1}(a') + \mu^n_{i,i+k}(b'e^{i\theta'}) \\ &= \mu^n_{i+1,i+k+1}(a') + \mu^n_{i+1,i+k+1}(b'e^{i\theta'}) = \mu^n_{i+1,i+k+1}(z).\end{aligned}$$

(The last step follows by using induction hypothesis N(1) on $\mu^n_{i+1,i+k+1}$.)

Property N(6): Suppose z is not a (non center) point in the i to $i + k + 1$ sized discs. Then $z \notin S$ and so $\mu^n_{i,i+k+1}(z) = \mu^n_{i+1,i+k+1}(z)$ by EQ 3.2. But z is also a (non center) point in the $i+1$ to $i+k+1$ sized discs and so by induction hypothesis N(6) on $\mu^n_{i+1,i+k+1}$, $\mu^n_{i+1,i+k+1}(z) = \nu_0(z)$ as desired.

Property N(7): The fact that this property holds is clear by construction.

Property N(1): If $a' \in K^n_{i+j} \cap I_n(0)$, $0 \leq j \leq k + 1$ and $b'e^{i\theta'}$ is such that $0 \leq b' \leq b_{i+j}$, then $z = a' + b'e^{i\theta'}$ is in S and $\mu^n_{i,i+k+1}(z)$ is defined via EQ 3.1-3.2 as $\mu^n_{i,i+k+1}(z) = \mu^n_{i+1,i+k+1}(a') + \mu^n_{i,i+k}(b'e^{i\theta'})$. But we've shown that N(4) and N(5) hold for $\mu^n_{i,i+k+1}$ and so, by N(5), since $a' \in I_n(0) - \bigcup\{T_i(a) - \{a\} \ : \ a \in K^n_i \text{ and } a \neq 0\}$, $\mu^n_{i,i+k+1}(a') = \mu^n_{i+1,i+k+1}(a')$. By N(4), since $b'e^{i\theta'} \in I_{i+k+1}(0)$, $\mu^n_{i,i+k+1}(b'e^{i\theta'}) = \mu^n_{i,i+k}(b'e^{i\theta'})$ and $\mu^n_{i,i+k+1}(z) = \mu^n_{i,i+k+1}(a') + \mu^n_{i,i+k+1}(b'e^{i\theta'})$ as desired. □

In summary:

Proposition 3.5. *The map $\mu^n_{i,i+k+1}$ defined via EQ 3.1 and EQ 3.2 is an embedding net with properties N(1)-N(7).*

Theorem 3.1. *For $\nu_n = \mu^n_{1,n}$ for all $n \in N$ and*

$$\psi(z) = \nu_n(z) \qquad \text{when} \qquad |z| \leq b_n,$$

ψ is an embedding, almost periodic net. It has the following properties:

1) For all $n \in N$, $a \in K_n$, $0 \leq b \leq b_n$ and $0 \leq \beta < 2\pi$,

$$\psi(a + be^{i\theta}) = \psi(a) + \psi(be^{i\theta}),$$

2) For any z, $z' \in R^2$ with $|z - z'| \geq 1$,

$$\left(\kappa - \sum_{j=1}^{\infty}\left(\frac{1}{4^j}\right)\right)|z - z'| \leq |\psi(z) - \psi(z')| \leq \left(\kappa' + \sum_{j=1}^{\infty}\left(\frac{1}{4^j}\right)\right)|z - z'|$$

3) $\psi = \nu_0$ on $R^2 - \cup_{n\in N} \cup_{a\in K_n - \{0\}} T_n(a)$,

In the next section we will investigate some of the properties of ψ, or, more specifically, some of the properties of h_ψ, the evaluation cocycle associated with ψ.

4. Properties of the Evaluation Cocycle of the Almost Periodic Spiral

In the following theorem we include the properties of the evaluation cocycle h_ψ of the almost periodic giant spiral ψ discussed thus far:

Theorem 4.1. *For (X_ψ, Z^2) and $h_\psi : X_\psi \times Z^2 \to R^2$*

1) (X_ψ, Z^2) is a minimal flow with X_ψ a compact metric space,

2) h_ψ is an embedding cocycle,

3) h_ψ is not close-to-linear.

Proof. Parts (1) and (2) follow from Corollary 2.1, Lemma 2.2, Proposition 2.1 and Theorem 3.1.

It remains to show that h_ψ is not close to linear. A complete proof of this fact is found in [M 94]. There we show that for any $L \in \mathcal{L}$ there exists a subset, $\{a_k\}_{k\in N}$, of Z^2 with $|a_k| \to \infty$ and $|h_\psi(\psi, a_k) - L(a_k)| \geq \epsilon |a_k|$ for some $\epsilon > 0$, and hence, by Theorem 1.3, h_ψ is not close to linear. A rough explanation of where the sequences $\{a_k\}_{k\in N}$ are found follows:

Let $L \in \mathcal{L}$. Suppose first that L is not the identity transformation with $L(e_1) = v$ and $|e_1 - v| > 0$. (A similar argument holds if only $|e_2 - L(e_2)| > 0$.) Consider b_ie_1 on the boundary of $I_i(0)$; we know that $\psi(b_ie_1) = \nu_0(b_ie_1) = b_ie_1$, and hence, if a_i is the largest integer less than b_i, $|\psi(a_ie_1) - a_ie_1| \leq \frac{1}{4^{i-1}}$. Thus, for I large enough,

$$\frac{|L(a_ie_1) - h_\psi(\psi, a_ie_1)|}{a_i} \geq |v - e_1| - \frac{1}{4^{I-1}} > 0.$$

In a similar manner, if L is the identity transformation, we choose $z_i \in \{z \in R^2 : |z| = \exp(e^{(2k_i+1)\pi})\}$, making sure z_i is not in any of the modified discs. Then $\psi(z_i) = \nu_0(z_i) = -z_i$ and the closest lattice point to z_i will provide our sequence. ∎

When (X, Z^m) is uniquely ergodic, all embedding cocycles are close to linear. The construction of the almost periodic spiral shows that this need not be true in general.

We conclude with a brief discussion of several questions which are raised with this construction. First, we note that the embedding cocycles studied prior to the construction of the almost periodic spiral, namely the embedding close to linear cocycles and the cocycles of Theorem 1.1, all yield suspensions that are time changes of the constant one suspension [KMS 94], [MM 95]. It is interesting to note that this is true for h_ψ as well [M 94]. An important question that remains open is whether this is true for every embedding cocycle. If this were the case, then all suspensions could be realized as the time change of a constant one suspension.

Another property of interest in the study of cocycles of Z^m flows is the property of nonsingularity. A cocycle $h = (h_1, \dots, h_m) : X \times Z^m \to R^m$ is said to be *nonsingular* if its collection of *invariant cocycle integrals*, given by

$$\left\{ \begin{pmatrix} \int_X h_1(x, e_1) d\mu_1 & \cdots & \int_X h_1(x, e_m) d\mu_m \\ \vdots & & \vdots \\ \int_X h_m(x, e_1) d\mu_1 & \cdots & \int_X h_m(x, e_m) d\mu_m \end{pmatrix} \right\}_{\{\mu_1, \dots, \mu_m\} \subset \mathcal{M}}$$

are all nonsingular. $\mathcal{M}$ refers to the set of invariant Borel probability measures on (X, Z^m) and $\{e_1, \dots, e_m\}$ refers to the standard basis.

When $m = 1$, the cocycles $h : X \times Z \to R$ correspond to the continuous functions $f : X \to R$ and in this case integration does yield information about individual cocycles. For example:

Theorem 4.2. *[FKMS 93] For $f : X \to R$ a continuous function, the cocycle h_f is covering if and only if $\int f d\mu \neq 0$ for every invariant Borel probability measure μ on X.*

The development of invariant cocycle integrals was motivated by the question of whether there is an appropriate higher dimensional analog to integration which would also yield information about the properties of individual cocycles. This question is addressed in [KMS2 94] and [M 94]. The higher dimensional analog of Theorem 4.2 is:

Conjecture 4.1. *Let $h : X \times Z^m \to R^m$ be a cocycle. Then h is covering if and only if h is nonsingular.*

In [M 94] it is shown that nonsingularity is a sufficient condition to imply covering. Based on the behavior of the examples of cocycles studied prior to the development of h_ψ, the necessity seemed reasonable as well. It is not difficult to verify that if $h = L + g$ is close to linear with $L \in \mathcal{L}$ and $g \in \overline{\mathcal{B}}$, then the collection of invariant cocycle integrals of h is $\{L\}$. Thus covering, close to linear cocycles are all nonsingular. (An unpublished example shows that the converse of this statement need not be true, so nonsingularity is a distinct characterization from covering, close to linearity.) The covering cocycles of Theorem 1.1 are also nonsingular [MM 95].

The evaluation cocycle h_ψ of the almost periodic giant spiral is embedding (and hence covering) but it is not nonsingular. The invariant cocycle integrals of h_ψ are

$$\left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} : a^2 + b^2 \leq 1,\ c^2 + d^2 \leq 1 \right\}.$$

Thus, this example seems to indicate that a characterization for covering behavior using integration does not exist for $m > 1$, at least not for this particular higher dimensional analog to integration.

For our final observation, we turn our attention to the two dynamical systems associated with any embedding cocycle $h : X \times Z^m \to R^m$, that is, the underlying Z^m flow (X, Z^m) and the R^m suspension (X_h, R^m). The suspension (X_{h_ψ}, R^2) of (X_ψ, Z^2) is isomorphic to a suspension (Y_g, R^2) built using a nonsingular cocycle $g : Y \times Z^2 \to R^2$ on a flow (Y, Z^2). So dramatically different cocycles can give rise to isomorphic suspensions. An interesting question is what this implies about the two underlying Z^2 flows. One can view the flows (X_ψ, Z^2) and (Y, Z^2) as being Kakutani equivalent in the topological sense. Further work is needed to determine whether this could be useful in providing a coarse classification of lattice actions of compact metric spaces.

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