The Structural Stability of Topological Cocycles

K. M. Madden
Drew University

N. G. Markley
Lehigh University

M. Sears
University of Witwatersrand

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Abstract

Cocycles of Z^m actions on compact metric spaces can be used to construct R^m actions or flows, called suspension flows. A suspension provides a higher dimensional analog to the familiar flow under a function and we look to this construction as a way of generating interesting R^m flows. Even more importantly, an R^m flow with a free dense orbit has an almost one to one extension which is a suspension [6] and thus suspensions can be used to model general R^m flows. In this paper we examine the sensitivity of the suspension construction to small perturbations in the cocycle. Theorem 4.7 establishes the fact that two cocycles that are sufficiently close yield suspensions that are isomorphic up to a time change.

1 Introduction

Let X be a compact metric space and let Z^m act as a group of commuting homeomorphisms on X. That is, we have a Z^m action on X, and we call (X, Z^m) a flow. For $a \in Z^m$, we denote the action of a on $x \in X$ by ax. A *cocycle* for the flow (X, Z^m) is a continuous map $h : X \times Z^m \to R^m$ satisfying the cocycle equation:

$$h(x, a + b) = h(x, a) + h(ax, b)$$

for all $a, b \in Z^m$ and $x \in X$.

A cocycle $h : X \times Z^m \to R^m$ can be used to construct the *suspension* (X_h, R^m) of the flow (X, Z^m). This is done as follows: for each $a \in Z^m$ we obtain a homeomorphism on $X \times R^m$ given by

$$T_a(x, v) = (ax, v - h(x, a))$$

for $a \in Z^m$, $x \in X$ and $v \in R^m$. Because h is a cocycle, it is easily checked that $T_{a+b} = T_a \circ T_b$. Hence, Z^m acts as a group of commuting homeomorphisms on $X \times R^m$, and we have a Z^m action on $X \times R^m$. We also have a natural R^m action on $X \times R^m$ given as follows: for each $w \in R^m$ we obtain a homeomorphism on $X \times R^m$ via

$$((x, v), w) \to (x, v + w)$$

which is continuous in all three variables and clearly defines an R^m action on $X \times R^m$.

We form

$$X_h = X \times R^m / \{T_a : a \in Z^m\},$$

the quotient space of $X \times R^m$ modulo the Z^m action on $X \times R^m$ furnished with the quotient topology. Let

$$\pi_h : X \times R^m \to X_h$$

be the usual projection. It is easily verified that the Z^m and R^m actions on $X \times R^m$ commute, and thus, R^m acts as a group of commuting homeomorphisms on X_h via

$$(\pi_h(x, v), w) \to \pi_h(x, v + w).$$

The flow (X_h, R^m) is referred to as the R^m suspension of (X, Z^m) built using the cocycle h. When $m = 1$, the cocycles $h : X \times Z \to R$ correspond to the continuous functions from X to R and the suspension flow is the familiar flow under a function. This construction is generally carried out with strictly positive valued functions because the topology of the resulting space can otherwise be less than desirable; there are examples that contain a point whose closure is the whole space. Using a strictly positive function will ensure that the resulting space is compact and metric and that (X, Z) occurs as a global section. When $m > 1$ it is not surprising that the topological properties of X_h continue to depend heavily on the properties of the cocycle h. We will restrict our attention to the *embedding cocycles*, cocycles whose suspensions are the topologically appropriate higher dimensional analog to the flow under a positive function; that is, a suspension built with an embedding cocycle will be compact and metric and will contain (X, Z^m) as a global section. We will say more about embedding cocycles at the end of this section.

It is natural to write our Z^m actions on the left and R^m actions on the right. A Z^m action {an R^m action} has a free orbit provided that for some x in the space $ax = x$ $\{xv = x\}$ only if $a = 0$ $\{v = 0\}$. The action itself is said to be free if every orbit is free. If for some $a \neq 0$ in Z^m we have $ax = x$ for all $x \in X$, then we do not really have a full Z^m action on X. Consequently it is not surprising that most of our results require a free dense orbit. This will be a standing assumption for the remainder of this work.

Suspensions are of interest in the study of R^m actions. We look to R^m suspensions as a way of generating examples of R^m flows with interesting dynamical behavior. Conversely, as the following theorem shows, suspensions play an important role in modeling general R^m flows [6].

Theorem 1.1 *An R^m flow (Y, R^m) on a compact metric space Y with a free dense orbit has an almost one-to-one extension which is a suspension.*

It is natural to ask how sensitive the suspension construction is to a small change in the cocycle. Analogous to the role played by the continuous function in the flow under a function, the cocycle h determines the return times of a point to the global section of (X, Z^m) embedded in (X_h, R^m). A small change in h would seem to result in a small change in these return times and hence in a new suspension that was a time change of the original one. We will say a cocycle $h : X \times Z^m \to R^m$ is *structurally stable* roughly when a small perturbation of h yields a suspension that is isomorphic to a time change of (X_h, R^m). That is, the orbit structure of a suspension built with a cocycle near h is the same as the orbit structure of (X_h, R^m). The goal of this work is to prove Theorem 4.7 which states that all embedding cocycles are structurally stable.

Structurally stability is discussed in greater detail in Section 2 and the problem of ascertaining when an embedding cocycle h is structurally stable is reduced to a problem of constructing a map $\psi : X \times R^m \to R^m$. This map sends cocycle images of h to cocycle images of h', a small perturbation of h, and satisfies a "cocycle like" equation. When the map is a homeomorphism for fixed x, it induces a homeomorphism $\Psi : X_h \to X_{h'}$ with $\Psi(\mathcal{O}(\pi_h(x,0))) = \mathcal{O}(\pi_{h'}(x,0))$.

The "geometry" of h will play a crucial role in the construction of ψ; associated with h and any point $x \in X$ is a tiling of R^m, namely the Voronoi tiling determined by the points $\{h(x,a) : a \in Z^m\}$, and in Section 3 we discuss these tilings further. The problem of constructing the map ψ then reduces to the problem of defining an appropriate map on the boundaries and centers of the Voronoi tiles. To construct ψ, we then extend this map linearly on the line segments in each tile extending radially between the tile center and the tile boundary. In Section 4, the map defined on the tile centers and boundaries is introduced and with it we are able to prove our main result, Theorem 4.7, that all embedding cocycles are structurally stable.

We close this section with a brief discussion of the embedding cocycles, the class of cocycles for which our results apply. The embedding cocycles are of interest because, by definition, a suspension constructed with an embedding cocycle is the appropriate higher dimensional analog to a flow under a positive function. These cocycles were studied in [1] and the results that follow can be found there.

Definition 1.2 *A cocycle $h : X \times Z^m \to R^m$ is embedding if*

(a) X_h is a Hausdorff space,

(b) The projection $\pi : X \times R^m \to X_h$ is one-to-one on $X \times \{v \in R^m : |v| < \varepsilon\}$ for some $\varepsilon > 0$. (i.e. X can be embedded in X_h as a global section of the flow.)

(The norm we will use in R^m is $|v| = \sum |v_i|$.)

We can say more about the topology of X_h when h is embedding:

Theorem 1.3 *Let $h : X \times Z^m \to R^m$ be embedding. Then X_h is compact.*

The proof of this theorem depends heavily on a salient feature of an embedding cocycle h, that is, it has "sufficient growth":

Theorem 1.4 *Let $h : X \times Z^m \to R^m$ be a cocycle. Then h is embedding if and only if there exists constants $B, B' > 0$ such that for all $a \in Z^m$ and $x \in X$,*

$$B|a| \leq |h(x,a)| \leq B'|a|.$$

Corollary 1.5 *Let $h : X \times Z^m \to R^m$ be a cocycle. Then h is embedding if and only if there exists constants $B, B' > 0$ such that for all $a, b \in Z^m$ and $x \in X$,*

$$B|a-b| \leq |h(x,a) - h(x,b)| \leq B'|a-b|.$$

Denote by $\mathcal{C}$ the collection of all R^m valued cocycles on (X, Z^m). Then

$$\begin{aligned} ||h|| &= \sup\left\{\frac{|h(x,a)|}{|a|} : x \in X, a \in Z^m, a \neq 0\right\} \\ &= \sup\left\{|h(x,e_i)| : x \in X, 1 \leq i \leq m\right\} \end{aligned}$$

defines a norm on $\mathcal{C}$ and with respect to this norm $\mathcal{C}$ is a separable Banach space. The embedding cocycles form an open subset of $\mathcal{C}$ and thus a sufficiently small perturbation of an embedding cocycle remains embedding.

2 Structural Stability

When h is an embedding cocycle, X_h contains a global section $\pi_h(X \times \{0\})$ which is a copy of X. The return locations to this global section are given by the Z^m action on X and the return times by the values of the cocycle h. Given two embedding cocycles h and g, a homeomorphism $\Psi : X_h \to X_g$ mapping orbits to orbits induces a map of the orbit space (X, Z^m) onto itself via the global sections $\pi_h(X \times \{0\})$ and $\pi_g(X \times \{0\})$. Denote by $\mathcal{O}(\pi_h(x,0))$ the orbit $\{\pi_h(x,v) : v \in R^m\}$ of $\pi_h(x,0)$ in (X_h, R^m). If $\Psi(\mathcal{O}(\pi_h(x,0))) = \mathcal{O}(\pi_g(x,0))$ for all $x \in X$ then the map induced by Ψ on the orbit space of (X, Z^m) is the identity and the role of (X, Z^m) as a global section is preserved as completely as possible; in this case, (X_h, R^m) is isomorphic to a time change of (X_g, R^m). Thus we are led to the following definition:

Definition 2.1 *Two embedding cocycles $h, g : X \times Z^m \to R^m$ are structurally equivalent if and only if there exists a homeomorphism $\Psi : X_h \to X_g$ such that*

$$\Psi(\mathcal{O}(\pi_h(x,0))) = \mathcal{O}(\pi_g(x,0))$$

for all $x \in X$.

It is natural to ask whether cocycles h and $h+p$ are structurally equivalent when $p : X \times Z^m \to R^m$ is a perturbation cocycle of small enough norm. That is, we ask whether h is structurally stable:

Definition 2.2 *An embedding cocycle $h : X \times Z^m \to R^m$ is structurally stable if and only if there exists $\varepsilon > 0$ such that when $||h - g|| < \varepsilon$, then h and g are structurally equivalent.*

The existence of a map $\Psi : X_h \to X_g$ as described in Definition 2.1 can be induced by a map from $X \times R^m$ to R^m:

Theorem 2.3 *Let h and g be embedding cocycles. Then h and g are structurally equivalent if there exists a continuous map $\psi : X \times R^m \to R^m$ satisfying:*

(a) For all $x \in X$ and $a \in Z^m$, $\psi(ax, v - h(x,a)) = \psi(x,v) - g(x,a)$,

(b) For all $x \in X$, the map $\psi(x, \cdot\,) : R^m \to R^m$ is a homeomorphism.

Furthermore, when all the orbits of (X, Z^m) are free, the converse is true.

Proof: Set $\tilde{\Psi} : X \times R^m \to X \times R^m$ by

$$\tilde{\Psi}(x,v) = (x, \psi(x,v)).$$

Note that

$$\tilde{\Psi}(ax, v - h(x,a)) = (ax, \psi(ax, v - h(x,a))) = (ax, \psi(x,v) - g(x,a))$$

and thus $\tilde{\Psi} \circ T_a = S_a \circ \tilde{\Psi}$ where $T_a = (ax, v - h(x,a))$ and $S_a = (ax, v - g(x,a))$ are the Z^m actions on $X \times R^m$ given by h and g respectively.

Thus $\tilde{\Psi}$ induces the required Ψ via

$$\Psi(\pi_h(x,v)) = \pi_g(\tilde{\Psi}(x,v)).$$

Conversely, we note that for a fixed x, the map $v \to \pi_g(x,v)$ is onto $\mathcal{O}(\pi_g(x,0))$, and when the action is free it is also one to one. Let $\Psi : X_h \to X_g$ be as given in Definition 2.1. We define $\tilde{\Psi} : X \times R^m \to X \times R^m$ by

$$\tilde{\Psi}(x,v) = \pi_g^{-1}(\Psi(\pi_h(x,v))) \cap (\{x\} \times R^m).$$

Clearly $\pi_g \circ \tilde{\Psi} = \Psi \circ \pi_h$ and $\tilde{\Psi}$ has the form

$$\tilde{\Psi}(x, v) = (x, \psi(x, v)).$$

$\tilde{\Psi} \circ T_a = S_a \circ \tilde{\Psi}$ because every action is free and $\tilde{\Psi}$ is continuous because π_g is a local homeomorphism. □

To show that a cocycle h is structurally stable we will need to be able to construct a map ψ as described in the previous theorem for every cocycle g within some ε neighborhood of h. This will be accomplished as follows:

Theorem 2.4 *An embedding cocycle h is structurally stable if there is a constant $M > 0$ depending only on h such that for every cocycle p there exists a continuous function $\Phi : X \times R^m \to R^m$ satisfying*

(a) For all $x \in X$ and $a \in Z^m$, $\Phi(ax, v - h(x, a)) = \Phi(x, v) - p(x, a)$,

(b) For all $x \in X$ and v, $w \in R^m$, $|\Phi(x, v) - \Phi(x, w)| \leq M\,||p||\,|v - w|$.

Proof: Let a cocycle p be given and suppose that Φ exists as described. We'll show that h and $h + p$ are structurally equivalent when $||p|| < \frac{1}{M}$. Define $\psi : X \times R^m \to R^m$ via

$$\psi(x, v) = v + \Phi(x, v).$$

Clearly ψ is continuous and for all $x \in X$ and $a \in Z^m$,

$$\begin{aligned} \psi(ax, v - h(x, a)) &= v - h(x, a) + \Phi(ax, v - h(x, a)) \\ &= v + \Phi(x, v) - (h(x, a) + p(x, a)) \\ &= \psi(x, v) - (h(x, a) + p(x, a)). \end{aligned}$$

It remains to show that $\psi(x, \cdot) : R^m \to R^m$ is a homeomorphism for all $x \in X$. It is one to one since for $|v - w| > 0$,

$$\begin{aligned} |\psi(x, v) - \psi(x, w)| &= |(v - w) - (\Phi(x, v) - \Phi(x, w))| \\ &\geq \Big| |(v - w)| - |(\Phi(x, v) - \Phi(x, w))| \Big| \\ &\geq (1 - M\,||p||)\,|v - w|. \end{aligned}$$

The fact that it is onto is now a consequence of

$$|\psi(x, v) - \psi(x, w)| \geq (1 - M\,||p||)\,|v - w|$$

and the Borsuk-Ulam Theorem (For similar proofs see [[4], p.117] or [[3], p 1926].) We see $\psi(x,\cdot) : R^m \to R^m$ has a continuous inverse similarly. Consider a sequence $\{v_n\}$ in R^m and a point w in R^m such that $\psi(x,v_n)$ converges to $\psi(x,w)$. Then

$$\begin{aligned} 0 &= \lim_{n\to\infty} |\psi(x,v_n) - \psi(x,w)| \\ &= \lim_{n\to\infty} |(v_n - w) - (\Phi(x,v_n) - \Phi(x,w))| \\ &\geq (1 - M\,||p||) \lim_{n\to\infty} |v_n - w| \end{aligned}$$

and v_n converges to w □

The construction of the map Φ will depend on the collection of tilings of R^m determined by h. These tilings are discussed in the next section.

3 The Cocycle Voronoi Tiling

There is a useful analogy between the construction of suspension flows and covering spaces. This analogy provided motivation for many of the ideas and proofs in [1]. In particular, it was shown that for embedding cocycles there is natural construction of a fundamental domain analogous to the construction of a fundamental domain for covering spaces when the covering transformations are all isometries. The following theorem summarizes these results from the last section of [1]:

Theorem 3.1 *The cocycle $h : X \times Z^m \to R^m$ is embedding if and only if there exists a compact subset F of $X \times R^m$ satisfying the following conditions:*

(a) $\pi(F) = X_h$,

(b) π *is one-to-one on the interior of* F,

(c) F *equals the closure of its interior,*

(d) $F_x = \{v \in R^m : (x,v) \in F\}$ *is convex for all* x *in* X,

(e) $X \times \{0\}$ *is contained in* F,

(f) the function $g : X \times S^{m-1} \to R$ defined by

$$g(x,u) = \sup\{t \geq 0 : (x,tu) \in F\}$$

is continuous and positive where $S^{m-1} = \{u \in R^m : |u| = 1\}$.

A subset F of $X \times R^m$ satisfying conditions (a) through (f) in the above theorem is called a *tiling domain* or a *Voronoi tiling domain* because for all $x \in X$,

$$\mathcal{T}_x = \{F_{ax} + h(x,a) : a \in Z^m\}$$

is a tiling of R^m by compact, convex polytopes where, as defined in the theorem,

$$F_x = \{v \in R^m : (x,v) \in F\}.$$

We'll refer to the tiling $\mathcal{T}_x$ as the tiling of R^m determined by $x \in X$. Given an embedding cocycle h, the existence of a tiling domain is very constructive, simply set

$$F = \{(x,v) : ||v|| \leq ||h(x,a) - v|| \text{ for all } a \in Z^m\}.$$

We observe that for each $x \in X$ the tiling $\mathcal{T}_x$ of R^m is the Voronoi tiling associated with the points $\{h(x,a) : a \in Z^m\}$. (Voronoi tilings, also known as Dirichlet, Thiessen or Wigner-Seitz tessellations, have found a wide variety of applications in science and mathematics. For more on Voronoi tilings, consult [2], [5].)

If F is a tiling domain, then in [[1], p. 444] it is shown that

$$\text{Int}(F) = \bigcup_{x \in X} \{x\} \times \text{Int}(F_x)$$

and

$$\partial F = \{(x, g(x,u)u) : x \in X \text{ and } ||u|| = 1\}.$$

Furthermore, there exists positive numbers β and κ such that every tile $F_{ax} + h(x,a)$ contains $B_\beta(h(x,a))$ and is contained in $B_\kappa(h(x,a))$. ($B_r(v)$ denotes the open ball of radius r around $v \in R^m$.) In particular, the distance from $h(x,a)$ to a point on the boundary of the tile $F_{ax} + h(x,a)$ is between β and κ. We will need several special consequences of the general properties of tiling domains and present them next as a series of remarks.

Remark 3.2 *Given $\varepsilon > 0$, there exists $r > 0$ such that for a, $b \in Z^m$, $|a-b| < r$ whenever the tiles $F_{ax} + h(x,a)$ and $F_{bx} + h(x,b)$ have boundary points v and v' (not necessarily distinct) satisfying $|v - v'| < \varepsilon$.*

Proof: For $a \neq b$,

$$\begin{aligned} B|a-b| &\leq |h(x,a) - h(x,b)| \\ &\leq |h(x,a) - v| + |v - v'| + |v' - h(x,b)| \leq 2\kappa + \varepsilon. \end{aligned}$$

(Recall B is the embedding cocycle constant given in Corollary 1.5.) □

Remark 3.3 *Let v and w be points on the same face D of the tile F_x. Then if θ is the angle between v and $v - w$,*

$$\sin\theta \geq \frac{\beta}{\kappa}.$$

Proof: First note that $B_\beta(0) \subset F_x \subset B_\kappa(0)$ and without loss of generality D is an $m-1$ dimensional face. Let L be the hyperplane containing D and note that L intersect $B_\beta(0)$ is empty. So the line L through v in the direction of $v - w$ does not intersect $B_\beta(0)$ and

$$\psi \leq \theta \leq 180^\circ - \psi$$

where ψ is the angle between the vector v and the line through the endpoint of v tangent to $B_\beta(0)$. Since the minimal value of $\sin\psi$ is β/κ when $||v|| = \kappa$, we have

$$\frac{\beta}{\kappa} \leq \sin\psi \leq \sin\theta.$$

□

Let F be a tiling domain for an embedding cocycle h and let $\mathcal{B}_x$ denote the set of boundary points of the tiles in the tiling $\mathcal{T}_x$. Set $\mathcal{B} = \{(x,v) : v \in \mathcal{B}_x\}$.

Remark 3.4 *Let F be the tiling domain for an embedding cocycle h. Then*

(a) $F_{bx} + h(x,b) - h(x,a) = F_{(b-a)ax} + h(ax, b-a)$

(b) $\mathcal{T}_x - h(x,a) = \mathcal{T}_{ax}$ for all $a \in Z^m$,

(c) $\mathcal{B}_x - h(x,a) = \mathcal{B}_{ax}$ for all $a \in Z^m$.

The following remark tells us that boundary points cannot "disappear" as we vary $x \in X$.

Remark 3.5 *Let $h : X \times Z^m \to R^m$ be an embedding cocycle and let F be its tiling domain. Then $\mathcal{B}$ is closed in $X \times R^m$.*

Proof: This follows immediately from

$$\partial F = \{(x, g(x,u)u) : x \in X \text{ and } ||u|| = 1\}$$

and the fact that g is continuous. □

The final remark also follows immediately from this description of ∂F and the continuity of g.

Remark 3.6 *If the sequence x_n converges to x in X and (x, v) is in $\mathcal{B}$, then there exists a sequence v_n in R converging to v such that (x_n, v_n) is in $\mathcal{B}$.*

Given an embedding cocycle h, our eventual goal is to construct the map Φ as described in Theorem 2.4 for any p. This can be achieved, as the following theorem demonstrates, by appropriately defining $\Phi(x, \cdot)$ on the boundaries and centers of the tiles in $\mathcal{T}_x$. It will be convenient to let $\mathcal{R}_x = \{h(x, a) : a \in Z^m\}$ and $\mathcal{R} = \{(x, v) : v \in \mathcal{R}_x\}$.

Theorem 3.7 *Let $h : X \times Z^m \to R^m$ be an embedding cocycle. Suppose there exists a constant $M > 0$ depending only on h such that for every cocycle p there exists a continuous map φ from $\mathcal{B} \cup \mathcal{R}$ into R^m satisfying:*

(a) For all $x \in X$, $v \in \mathcal{B}_x$ and $a \in Z^m$, $\varphi(ax, v - h(x, a)) = \varphi(x, v) - p(x, a)$,

(b) For all $x \in X$ and $v \in \mathcal{B}_x$ a boundary point of the tile centered at $h(x, a)$,

$$|\varphi(x, v) - \varphi(x, h(x, a))| \leq M\, ||p||\, |v - h(x, a)|$$

(c) For all $x \in X$ and v, $w \in \mathcal{B}_x$ boundary points of the tile centered at $h(x, a)$,

$$|\varphi(x, v) - \varphi(x, w)| \leq M\, ||p||\, |v - w|.$$

Then the map Φ as described in Theorem 2.4 exists and h is structurally stable.

Proof: For $(x,v) \in X \times R^m$ define $\Phi(x,v)$ as follows: Suppose v is in the Voronoi tile in $\mathcal{T}_x$ centered at $h(x,b)$. Then $v = h(x,b) + t(u - h(x,b))$ where $u \in \mathcal{B}_x$ and $0 \leq t \leq 1$. Define

$$\Phi(x,v) = \varphi(x, h(x,b)) + t(\varphi(x,u) - \varphi(x, h(x,b))).$$

In other words, $\varphi(x,\,\cdot\,)$ is defined on tile centers and boundaries, and we obtain $\Phi(x,\,\cdot\,)$ by extending linearly along radial line segments from tile centers.

It is clear that Φ is continuous. Using Remark 3.4, we also see that for $v = h(x,b) + t(u - h(x,b))$ in the tile of $\mathcal{T}_x$ centered at $h(x,b)$ and u on its boundary,

$$\begin{aligned} v - h(x,a) &= h(x,b) - h(x,a) + t(u - h(x,a) + h(x,a) - h(x,b)) \\ &= h(ax, b-a) + t((u - h(x,a)) - h(ax, b-a)) \end{aligned}$$

is in the tile of $\mathcal{T}_{ax}$ centered at $h(ax, b-a)$ and $u - h(x,a)$ is on its boundary. Property (a) of Theorem 2.4 now follows by a routine calculation which we omit.

The fact that Φ also satisfies property (b) is more difficult and depends on the geometric properties of the Voronoi tilings associated with the embedding cocycle h. First suppose that $u = h(x,b) + t(v - h(x,b))$ and $u' = h(x,b) + s(w - h(x,b))$ where $0 \leq s,t \leq 1$ and where v and w are on the same k-dimensional face F of the tile in $\mathcal{T}_x$ centered at $h(x,b)$. If $s = t$ it is clear that

$$\begin{aligned} &|\Phi(x,u) - \Phi(x,u')| \\ &= |\varphi(x, h(x,b) + t(\varphi(x,v) - \varphi(x,h(x,b)) - \varphi(x,h(x,b)) \\ &- t(\varphi(x,w) + \varphi(x,h(x,b))| \\ &= t|\varphi(x,v) - \varphi(x,w)| \leq tM||p||\,|v-w| = M||p||\,|u-u'|. \end{aligned}$$

So, without loss of generality, assume that $s < t$, say $t = s + \delta$. This is illustrated below:

Figure 1

By Remark 3.3, the angle θ between $v - h(x,b)$ and $v - w$ satisfies:

$$1 \leq \frac{1}{\sin(\theta)} \leq \frac{\kappa}{\beta}.$$

Using the Law of Sines,

$$\frac{\delta|v-h(x,b)|}{\sin(\gamma)} = \frac{|t(v-h(x,b))-s(w-h(x,b))|}{\sin(\theta)}$$

and

$$\delta|v-h(x,b)| \leq \frac{\kappa}{\beta}\,|t(v-h(x,b))-s(w-h(x,b))|.$$

Similarly,

$$s|v-w| \leq \frac{\kappa}{\beta}\,|t(v-h(x,b))-s(w-h(x,b))|.$$

Thus

$$\begin{aligned}
|\Phi(x,u)-\Phi(x,u')| &= |t(\varphi(x,v)-\varphi(x,h(x,b))-s(\varphi(x,w)-\varphi(x,h(x,b))| \\
&= |s(\varphi(x,v)-\varphi(x,w))+\delta(\varphi(x,v)-\varphi(x,h(x,b)))| \\
&\leq s|\varphi(x,v)-\varphi(x,w)|+\delta|\varphi(x,v)-\varphi(x,h(x,b))| \\
&\leq sM||p||\,|v-w|+\delta M||p||\,|v-h(x,b)| \\
&\leq 2M\left(\frac{\kappa}{\beta}\right)||p||\,|t(v-h(x,b))-s(w-h(x,b))| \\
&= 2M\left(\frac{\kappa}{\beta}\right)||p||\,|u-u'|
\end{aligned}$$

and property (b) holds when $u = h(x,b)+t(v-h(x,b))$ and $u' = h(x,b)+s(w-h(x,b))$ where $0 \leq s,t \leq 1$ and where v and w are on the same k-dimensional face F of the tile in $\mathcal{T}_x$ centered at $h(x,b)$.

Now let u and u' be in the same tile in $\mathcal{T}_x$ centered at $h(x,b)$. Suppose the line segment from u to u' does not pass through $h(x,b)$. Then we can find points $\{u_0 = u, u_1, \ldots, u_n = u'\}$ on the line segment joining u and u' such that $u_i = h(x,b)+t_i(v_i+h(x,b))$ and v_i and v_{i+1} are on the same face of the tile in $\mathcal{T}_x$ centered at $h(x,b)$. (The line segment between u and u' is contained in the tile in $\mathcal{T}_x$ centered at $h(x,b)$ by convexity.) Then

$$\begin{aligned}
|\Phi(x,u)-\Phi(x,u')| &\leq \sum_{i=0}^{n-1}|\Phi(x,u_i)-\Phi(x,u_{i+1})| \\
&\leq 2M\left(\frac{\kappa}{\beta}\right)||p||\sum_{i=0}^{n-1}|u_i-u_{i+1}| \\
&= 2M\left(\frac{\kappa}{\beta}\right)||p|||u-u'|.
\end{aligned}$$

When $h(x,b)$ is on this line segment, the same argument works with $\{u, h(x,b), u'\}$. So property (b) holds when u and u' are in the same tile.

Similarly, let u and u' be arbitrary points in R^m and let $\{u_0 = u, u_1, \ldots, u_n = u'\}$ be points on the line segment joining u and u' such that u_i and u_{i+1} are in the same tile. Then

$$\begin{aligned} |\Phi(x,u) - \Phi(x,u')| &\leq \sum_{i=0}^{n-1} |\Phi(x,u_i) - \Phi(x,u_{i+1})| \\ &\leq 2M\left(\frac{\kappa}{\beta}\right) ||p|| \sum_{i=0}^{n-1} |u_i - u_{i+1}| \\ &= 2M\left(\frac{\kappa}{\beta}\right) ||p|| |u - u'| \end{aligned}$$

as desired. □

Thus the problem of determining that an embedding cocycle is structurally stable reduces to the problem of finding a constant $M > 0$ that depends only on h and, given an arbitrary cocycle p, a map φ as described in Theorem 3.7. The constant M and the procedure for constructing the maps φ is described in the next section.

4 The Boundary Weighting Function

When defining the function φ of Theorem 3.7, we need to define the image of the points in $\mathcal{R} \cup \mathcal{B}$ under φ. Each point $h(x,a)$ will be mapped by φ to $p(x,a)$. The image of a point $(x,v) \in \mathcal{B}$ is more complicated; ideally we would define $\varphi(x,v)$ to be the average of the cocycle images centered in tiles containing v as a boundary point. However, in doing this, φ may not be continuous; a boundary point cannot "disappear" in the limit but the number of tiles on which it lies may change. An example of this is illustrated below:

Figure 2

Thus, we will define $\varphi(x,v)$ to be a weighted average of the cocycle images "near" v. The weighting function will identify cocycle images whose tile boundaries are getting close to v. We next describe the construction of such a function, which we will call the boundary weighting function, and some of its properties.

First we define $\lambda : R \times R \to [0,1]$ via

$$\lambda(v, v') = \begin{cases} 1 - |v - v'| & \text{if } |v - v'| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly λ is continuous and invariant under translation. The function λ should be thought of as a warning light which indicates when some exotic behavior of the boundaries of the tiles might occur.

We now define a function $w : \mathcal{B} \times Z^m \to [0,1]$ called *the boundary weighting function* by setting

$w(x, v, a) = \sup \{\lambda(v, v') : v' \text{ is a boundary point of } F_{ax} + h(x, a) \in \mathcal{T}_x\}$.

Observe first that $w(x, v, a) = \lambda(v, v')$ for some $v' \in \mathcal{B}_x$ where v' is a boundary point of $F_{ax} + h(x, a)$. Secondly, as a consequence of Remark 3.4 we have

$$w(x, v, a + b) = w(ax, v - h(x, a), b).$$

Remark 4.1 *The function $w(x, v, a)$ has the following properties:*

(a) $w(x, v, a) = 1$ *if and only if v is on the boundary of the tile $F_{ax} + h(x, a)$ in $\mathcal{T}_x$*

(b) $w(x, v, a) = 0$ *if and only if v is a distance 1 or greater from the boundary of the tile $F_{ax} + h(x, a)$.*

(c) $0 < w(x, v, a) < 1$ *if and only if v is not on the boundary of the tile $F_{ax} + h(x, a)$ but it is within 1 of this tile.*

Theorem 4.2 *The function $w : \mathcal{B} \times Z^m \to [0,1]$ is continuous.*

Proof: Without loss of generality it suffices to fix a in Z^m and consider (x_n, v_n) converging to (x, v) in $\mathcal{B}$. For each $n \in N$ there exists $v'_n \in \mathcal{B}_{x_n}$ which is a boundary point of the tile $F_{ax_n} + h(x_n, a)$ such that $w(x_n, v_n, a) = \lambda(v_n, v'_n)$. Taking a subsequence if necessary, $v'_n \to v' \in \partial F_{ax} + h(x, a)$. Then $|v_n - v'_n| \to |v - v'|$ and

$$w(x_n, v_n, a) = \lambda(v_n, v'_n) \to \lambda(v, v') \leq w(x, v, a).$$

If, however, $w(x,v,a) = \lambda(v,w) > \lambda(v,v')$ for some $w \in \partial F_{ax} + h(x,a)$, then by Remark 3.6 we may choose $w_n \in \partial F_{ax_n}$ with $w_n \to w$. But then we have

$$\lim_{n\to\infty} w(x_n, v_n, a) = \lambda(v,v') < \lambda(v,w) = \lim_{n\to\infty} \lambda(v_n, w_n) \le \lim_{n\to\infty} w(x_n, v_n, a),$$

a contradiction.

So $\lim_{n\to\infty} w(x_n, v_n, a) = w(x,v,a)$ as desired. □

Theorem 4.3 *If v_1 and v_2 are in $\mathcal{B}_x$, then*

$$|w(x, v_1, a) - w(x, v_2, a)| \le |v_1 - v_2|$$

for all $a \in Z^m$.

Proof: We start with the three cases in Remark 4.1 for $w(x, v_1, a)$ and split the third into two cases according to whether v_2 is or is not the point in the boundary of $F_{ax} + h(x,a)$ determining the value of $w(x, v_1, a)$. Specifically we consider the following four possibilities for $w(x, v_1, a)$:

CASE 1: For all the boundary points w of the tile $F_{ax} + h(x,a)$ in $\mathcal{T}_x$ we have $|v_1 - w| > 1$ or equivalently $w(x, v_1, a) = 0$.

CASE 2: The point v_1 is a boundary point of the tile $F_{ax} + h(x,a)$ or equivalently $w(x, v_1, a) = 1$.

CASE 3: The point v_1 is not a boundary point of the tile $F_{ax} + h(x,a)$, and v_2 is the closest boundary point of $F_{ax} + h(x,a)$ to v_1. Furthermore, $|v_1 - v_2| < 1$ and $w(x, v_1, a) = 1 - |v_1 - v_2|$.

CASE 4: The point v_1 is not a boundary point of the tile $F_{ax} + h(x,a)$, and v_2 is not the closest boundary point of $F_{ax} + h(x,a)$ to v_1. Furthermore, $w(x, v_1, a) = 1 - |v_1 - u|$ where u is a boundary point of $F_{ax} + h(x,a)$ within 1 of v_1.

There are a similar four possibilities for $w(x, v_2, a)$ and thus, sixteen total possibilities (many easily discarded) must be checked.

If $w(x, v_1, a) = 0 = w(x, v_2, a)$ or $w(x, v_1, a) = 1 = w(x, v_2, a)$, then the claim is clear. Similarly if $w(x, v_1, a) = 1$ and $w(x, v_2, a) = 1 - |v_1 - v_2|$, or vice versa, the claim is equally clear.

Next suppose $w(x, v_1, a) = 0$ and $w(x, v_2, a) = 1$. In this case we know v_2 is a boundary point of $F_{ax} + h(x,a)$ and v_1 is further than 1 from all boundary points of $F_{ax} + h(x,a)$ and thus

$$|v_1 - v_2| > 1 = |w(x, v_1, a) - w(x, v_2, a)|$$

The case of $w(x, v_1, a) = 1$ and $w(x, v_2, a) = 0$ is handled similarly.

The cases when $w(x, v_2, a) = 0$ and v_1 satisfies Case 3 (or vice versa) can not occur since the value of $w(x, v_2, a) = 0$ implies v_2 is not a boundary point of $F_{ax} + h(x, a)$. In fact, when v_1 satisfies Case 3, $w(x, v_2, a)$ must equal 1, allowing us to rule out three more cases. We have now dealt with 11 of the 16 cases.

In the final case with $w(x, v_1, a) = 0$, we have $w(x, v_2, a) = 1 - |v_2 - u|$, where u is a boundary point of $F_{ax} + h(x, a)$ and $|v_1 - u| > 1$ while $|v_2 - u| < 1$. By the triangle inequality $1 < |v_1 - u| \leq |v_1 - v_2| + |v_2 - u|$. Thus $1 - |v_2 - u| \leq |v_1 - v_2|$ and

$$|w(x, v_1, a) - w(x, v_2, a)| = 1 - |v_2 - u| \leq |v_1 - v_2|.$$

The reverse case of $w(x, v_1, a) = 1 - |v_1 - u|$ and $w(x, v_2, a) = 0$ is handled similarly.

In the final case with $w(x, v_1, a) = 1$, we have $w(x, v_2, a) = 1 - |v_2 - u|$. In this case, both v_1 and u are boundary points of $F_{ax} + h(x, a)$ but u is closer to v_2 than v_1 is. Thus

$$\begin{aligned} |w(x, v_1, a) - w(x, v_2, a)| &= |1 - 1 - |v_2 - u|\,| \\ &= |v_2 - u| \leq |v_2 - v_1|. \end{aligned}$$

As usual the reverse case is handled similarly.

Finally suppose both v_1 and v_2 belong in Case 4, so $w(x, v_i, a) = 1 - |v_i - u_i|$. In this case, u_1 and u_2 are boundary points of $F_{ax} + h(x, a)$ but v_1 and v_2 are not. Furthermore, $|v_1 - u_1| \leq |v_1 - u_2|$ and $|v_2 - u_2| \leq |v_2 - u_1|$. Without loss of generality, assume $|v_1 - u_1| \leq |v_2 - u_2|$. Then,

$$0 \leq |v_2 - u_2| - |v_1 - u_1| \leq |v_2 - u_1| - |v_1 - u_1| \leq |v_2 - v_1|$$

and

$$\begin{aligned} |w(x, v_1, a) - w(x, v_2, a)| &= |1 - |v_1 - u_1| - (1 - |v_2 - u_2|)| \\ &= |v_2 - u_2| - |v_1 - u_1| \\ &\leq |v_1 - v_2| \end{aligned}$$

as desired. □

Let us summarize the results of this section thus far; we have defined a continuous boundary weighting function $w : \mathcal{B} \times Z^m \to [0,1]$ which has the property that for points v_1, v_2 in $\mathcal{B}_x$, $|w(x, v_1, a) - w(x, v_2, a)| \leq |v_1 - v_2|$. The number $w(x, v, a)$ measures the "importance" of the cocycle image $h(x, a)$ to the point $v \in \mathcal{B}_x$ in the following sense: when $w(x, v, a) = 1$, v is a boundary point of the tile $F_{ax} + h(x, a)$, and when $w(x, v, a) = 0$, the tile $F_{ax} + h(x, a)$ is very far from v. For $0 < w(x, v, a) < 1$, v is a not a boundary point of the tile $F_{ax} + h(x, a)$ but it is close to a point $v' \in \partial F_{ax} + h(x, a)$ indicating a potential change in the configuration of the tiles for nearby x .

We will now use the boundary weighting function to define $\varphi(x, v)$ on $\mathcal{R} \cup \mathcal{B}$ as an appropriately weighted average of cocycle images. Let $p : X \times Z^m \to R^m$ be a cocycle. First for all $x \in X$ define

$$\varphi(x, h(x, a)) = p(x, a)$$

for $a \in Z^m$. Since the values of φ on $\mathcal{B}$ will be averages, it is convenient to define the normalizing function

$$W(x, v) = \sum_{b \in Z^m} w(x, v, b)$$

and note using the equation preceding Theorem 4.2 that $W(ax, v - h(x, a)) = W(x, v)$. Now define φ on $\mathcal{B}$ by

$$\varphi(x, v) = \frac{1}{W(x, v)} \sum_{b \in Z^m} w(x, v, b)\, p(x, b)$$

which is obviously a continuous function by Theorem 4.2. The following sequence of 3 lemmas show that the map φ satisfies the hypotheses (a), (b), and (c) respectively of Theorem 3.7.

Lemma 4.4 *For all $x \in X$, we know that $v \in \mathcal{B}_x$ and $a \in Z^m$,*

$$\varphi(x, v) = \varphi(ax, v - h(x, a)) + p(x, a).$$

Proof: For $v \in \mathcal{B}_x$, $v - h(x, a) \in \mathcal{B}_{ax}$ and

$$\begin{aligned} &\varphi(ax, v - h(x, a)) \\ &= \frac{1}{W(ax, v - h(x, a))} \sum_{b \in Z^m} w(ax, v - h(x, a), b)\, p(ax, b) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{W(x,v)} \sum_{b\in Z^m} w(x,v,a+b)\,(p(x,a+b)-p(x,a)) \\
&= \left(\frac{1}{W(x,v)} \sum_{b\in Z^m} w(x,v,a+b)\,p(x,a+b)\right) - p(x,a) \\
&= \varphi(x,v) - p(x,a)
\end{aligned}
$$

as desired. □

Lemma 4.5 *Let $v \in \mathcal{B}_x$ be boundary point of the tile $F_{bx} + h(x,b)$ in $\mathcal{T}_x$. Then there exists $M > 0$ which depends only on h such that*

$$|\varphi(x,v) - \varphi(x,h(x,b))| \le M\,||p||\,|v - h(x,b)|.$$

Proof: It follows from Remark 3.2 that $w(x,v,a) = 0$ when $|a-b| \ge r$. Consequently

$$
\begin{aligned}
&|\varphi(x,v) - \varphi(x,h(x,b))| \\
&\quad = \left|\left(\frac{1}{W(x,v)} \sum_{a\in Z^m} w(x,v,a)\,p(x,a)\right) - p(x,b)\right| \\
&\quad = \left|\sum_{a\in Z^m} \frac{w(x,v,a)}{W(x,v)}\,(p(x,a)-p(x,b))\right| \\
&\quad = \left|\sum_{\{a\in Z^m : |a-b|\le r\}} \frac{w(x,v,a)}{W(x,v)}\,(p(x,a)-p(x,b))\right| \\
&\quad \le \sum_{\{a\in Z^m : |a-b|\le r\}} |p(x,a)-p(x,b)| \\
&\quad = \sum_{\{a\in Z^m : |a-b|\le r\}} |p(ax,b-a)| \\
&\quad \le \sum_{\{a\in Z^m : |a-b|\le r\}} ||p||\,|b-a| \\
&\quad \le (2r+1)^m r ||p||.
\end{aligned}
$$

But $\beta \le |v - h(x,b)|$ so

$$|\varphi(x,v) - \varphi(x,h(x,b))| \le (2r+1)^m r ||p|| \le \frac{(2r+1)^m r}{\beta} ||p||\,|v-h(x,b)|$$

as desired. □

Lemma 4.6 *Let $\{v_1, v_2\} \subset \mathcal{B}_x$ be boundary points of the $F_{bx} + h(x, b)$ tile in $\mathcal{T}_x$. Then there exists $M > 0$ which depends only on h such that*

$$|\varphi(x, v_1) - \varphi(x, v_2)| \leq M\, ||p||\, |v_1 - v_2|.$$

Proof: First note that by Remark 3.2 and Theorem 4.3

$$\begin{aligned} |W(x, v_1) - W(x, v_2)| &\leq \sum_{\{a \in Z^m : |a| \leq r\}} |w(x, v_1, a) - w(x, v_2, a)| \\ &\leq \sum_{\{a \in Z^m : |a| \leq r\}} |v_1 - v_2| = (2r + 1)^m |v_1 - v_2|. \end{aligned}$$

Using $W(x, v) \geq 2$ and $p(x, a) - p(x, b) = p(bx, a - b)$ we get

$$\begin{aligned} &|\varphi(x, v_1) - \varphi(x, v_2)| \\ &= \left| \frac{1}{W(x, v_1)} \sum_{a \in Z^m} w(x, v_1, a)\, p(x, a) - \frac{1}{W(x, v_2)} \sum_{a \in Z^m} w(x, v_2, a)\, p(x, a) \right| \\ &= \left| \frac{1}{W(x, v_1)} \sum_{a \in Z^m} w(x, v_1, a)\, p(bx, a - b) - \frac{1}{W(x, v_2)} \sum_{a \in Z^m} w(x, v_2, a)\, p(bx, a - b) \right| \\ &\leq \frac{1}{W(x, v_1)} \sum_{a \in Z^m} |w(x, v_1, a) - w(x, v_2, a)|\ |p(bx, a - b)| + \\ &\qquad \left| \frac{1}{W(x, v_1)} - \frac{1}{W(x, v_2)} \right| \sum_{a \in Z^m} |w(x, v_2, a)|\ |p(bx, a - b)| \\ &\leq \sum_{\{a \in Z^m : |a| \leq r\}} |v_1 - v_2|\, ||p||\, |a - b| + (2r + 1)^m |v_1 - v_2| \sum_{\{a \in Z^m : |a| \leq r\}} ||p||\, |a - b| \\ &\leq \frac{|v_1 - v_2|}{\varepsilon} ||p||\, r \left((2r + 1)^m + (2r + 1)^{2m} \right) \\ &\leq M\, ||p||\, |v_1 - v_2| \end{aligned}$$

as desired. □

Theorem 4.7 *Let $h : X \times Z^m \to R^m$ be an embedding cocycle. Then h is structurally stable.*

Proof: Follows from Theorem 3.7 and Lemmas 4.4 - 4.6. □

Corollary 4.8 *If h and g are cocycles in the same component of the open set of embedding cocycles in $\mathcal{C}$, then h and g are structurally equivalent and the spaces X_h and X_g are homeomorphic. In particular, all embedding cocycles in the component of the identity cocycle $I(x, a) = a$ are time changes of its suspension flow (X_I, R^m)*

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